Semiregular automorphisms of vertex-transitive cubic graphs

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Abstract

An old conjecture of Marušič, Jordan and Klin asserts that any finite vertex-transitive graph has a non-trivial semiregular automorphism. Marušič and Scapellato proved this for cubic graphs. For these graphs, we make a stronger conjecture, to the effect that there is a semiregular automorphism of order tending to infinity with n. We prove that there is one of order greater than 2.

Key words: vertex-transitive graph, semiregular automorphism

A permutation σ is *semiregular* if all its cycles have the same length. An old conjecture made independently by Marušič, Jordan and Klin (see the introduction to [3] for details) asserts that any finite vertex-transitive graph has a non-trivial semiregular automorphism. Clearly there is no loss of generality in assuming that the graph is connected. Marušič and Scapellato proved:

Theorem 1 *A vertex-transitive connected cubic simple graph has a non-trivial semiregular automorphism.*

We need to reproduce the proof since we will use parts of it later.

PROOF. We argue by contradiction. Let G be a connected cubic vertex-transitive graph, and suppose that G has no non-trivial semiregular automorphism.

We first observe that, if σ is an automorphism of prime order greater than 3, then σ is semiregular. For, if σ fixes a vertex v, then it must fix the three

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neighbours of v, and then their neighbours, ad so on; since G is connected, we would find that σ is the identity, a contradiction.

So $|\operatorname{Aut}(G)| = 2^x 3^y$ for some x, y.

Next we show that $y \neq 0$. For suppose that $y = 0$. Then $\operatorname{Aut}(G)$ is a 2-group, so there is a non-identity element σ in its centre; and since $\operatorname{Aut}(G)$ is transitive, σ is semiregular, contrary to assumption.

Now it follows that G is arc-transitive, that is, $\operatorname{Aut}(G)$ is transitive on ordered pairs of adjacent vertices. For let σ be an automorphism of order 3. Then σ fixes a vertex. Arguing as before, there must be a vertex v such that σ fixes v and permutes its three neighbours transitively. Since G is vertex-transitive (by assumption), it is thus arc-transitive.

By Burnside's $p^\alpha q^\beta$-theorem, $\operatorname{Aut}(G)$ is soluble. So a minimal normal subgroup N of $\operatorname{Aut}(G)$ is elementary abelian. We split the argument into two cases, according as N is a 3-group or a 2-group.

Case 1: N is a 3-group. Since N is abelian, it acts regularly on each or its orbits. We further subdivide into cases as follows. Consider the stabiliser N_v of a vertex v.

Case 1A: N_v fixes the three neighbours of v. Since N is a normal subgroup of $\operatorname{Aut}(G)$, this holds for all vertices v. As before, we find that $N_v = 1$, so that N is semiregular, as of course are all its non-identity elements.

Case 1B: N_v permutes the three neighbours of v transitively. Then these neighbours lie in the same N-orbit, say O_1. Let v lie in the orbit O_2. Then N_v fixes O_2 pointwise (since N acts regularly on O_2) but acts semiregularly on O_1 as a group of order 3. So $O_1 \neq O_2$. Moreover, all edges from vertices in O_1 go to vertices in O_2, so G is bipartite, with bipartite blocks O_1 and O_2. Thus G is the complete bipartite graph $K_{3,3}$. But this graph has a regular automorphism of order 6.

Case 2: N is a 2-group. Choose a vertex v. Then $\operatorname{Aut}(G)_v$ acts transitively on the three neighbours of v, hence as either the symmetric group S_3 or the cyclic group A_3. So its normal subgroup N_v acts as S_3, A_3 or the trivial group. Since N is a 2-group, the first two cases are impossible, and N_v fixes the neighbours of v. Then the usual argument shows that N_v is trivial, so N is semiregular, and we are done. □

The proof allows the possibility that the semiregular element has order 2 or 3. However, in all the examples known to us, there are semiregular elements with order at least 4. We make the following conjecture:

Conjecture 2 *There is a function f, so that $f(n) \to \infty$ as $n \to \infty$, with the property that a connected vertex-transitive cubic graph on n vertices has a semiregular automorphism of order at least $f(n)$.*

In the rest of this paper, we show that there is always a semiregular automorphism of order at least 3, and end with some examples which give an upper bound to the growth of such a function.

Theorem 3 *Let G be a connected vertex-transitive cubic graph. Then G has a semiregular automorphism of order greater than 2.*

The proof depends on the following group-theoretic lemma.

Lemma 4 *If P is a 2-group which is not elementary abelian, and Q a core-free subgroup of P, then there is an element of P of order 4 whose square lies in no conjugate of Q.*

PROOF. Recall that a subgroup Q of P is *core-free* if Q contains no non-trivial normal subgroup of P. Note that, if Q is core-free, then $Q \cap \zeta(P) = 1$, where $\zeta(P)$ is the centre of P.

Our proof is by induction on $|P|$. Let P be a minimal counterexample. Let Q be a core-free subgroup of P such that

$$\mathcal{P}^2 = \{g^2 | g \in G\} \subseteq \bigcup_{g \in P} Q^g.$$

If the exponent of $\zeta(P)$ is at least 4, then the centre of P contains a square, and the proposition is clear, so assume $\zeta(P)$ is elementary abelian.

Let Z be a central subgroup of order 2 and consider P/Z and QZ/Z. If P/Z is elementary abelian then $1 \neq g^2 \in Z$ for some $g \in P$. In particular, as Q is core-free, $g^2 \notin Q$. So, we may as well assume that P/Z is not elementary abelian.

Now, $|P/Z| < |P|$ and $\{g^2 Z | g \in P\} \subseteq \bigcup_{g \in P} (QZ/Z)^g$. So, by the induction hypothesis, QZ/Z is not core-free in P/Z.

Set $N = \bigcap_{g \in P} (QZ)^g$. Now, $Z < N \leq QZ$, hence $NQ = ZQ$. In particular $\bigcap_{g \in P} (QN)^g = N$. Therefore QN/N is a core-free subgroup of P/N. Moreover $\{g^2 N | g \in P\} \subseteq \bigcup_{g \in P} (QN/N)^g$. So, by the induction hypothesis, P/N is elementary abelian, so that $\Phi(P) \subseteq N$, where $\Phi(P)$ is the Frattini subgroup of P.

Now $NQ = QZ$, $|Z| = 2$ and $Z < N$; so $N \cap Q$ has index 2 in N. Furthermore, $Q \cap N$ is core-free, therefore N is a subdirect product of copies of the cyclic group of order 2, and hence is a normal elementary abelian 2-subgroup of P. Since $\Phi(P) \subseteq N$, this implies that $\Phi(P)$ is elementary abelian.

Let Z_1 be another subgroup of $\zeta(P)$ of order 2. Applying the same argument, we get $\Phi(P) \subseteq QZ_1$ as well as $\Phi(P) \subseteq QZ$. If $QZ \neq QZ_1$ then $Q = QZ \cap QZ_1$. Therefore Q is a normal subgroup of P, a contradiction. Therefore $QZ = QZ_1$. This proves that $QZ = Q\zeta(P)$ (since $\zeta(P)$ is elementary abelian). In particular, as Q is core-free, $\zeta(P)$ has order 2, $P' = \zeta(P)$ and P has nilpotency class 2.

Let g, h be elements of P, $1 = [g, h]^2 = [g^2, h]$. Therefore g^2 lies in the centre of P. Thence $\mathcal{P}^2 \subseteq \zeta(P)$. So, either $\mathcal{P}^2 = 1$ or $\zeta(P)$ contains a square. In the former case P is elementary abelian, a contradiction. In the latter case P is not a counterexample. This concludes the proof. □

This is equivalent to the following result about permutation groups:

Corollary 5 *Let P be a transitive 2-group which is not elementary abelian. Then P contains a semiregular element of order 4.* □

This follows immediately from the lemma, on taking Q to be the stabiliser of a point in P. (An element of order 4 is semiregular if and only if its square has no fixed points.)

Now we can prove the Theorem. Let G be a vertex-transitive cubic graph which has no semiregular automorphism of order greater than 2. As in the proof of Theorem 1, $\mathrm{Aut}(G)$ has order divisible by the primes 2 and 3 only.

Suppose that $P = \mathrm{Aut}(G)$ is a 2-group. By the above Corollary, it is elementary abelian and regular, and G is a Cayley graph for P. Since G is a cubic graph, P is generated by three elements. Thus P has order 4 or 8, and G is K_4 or the cube; but each of these graphs has a semiregular automorphism of order 4. So 3 must divide $|\,\mathrm{Aut}(G)|$.

If 3 does not divide the order of the vertex stabiliser, then an element of order 3 is semiregular. (Note that in this case we cannot construct a semiregular automorphism of order greater than 3; but such an automorphism will exist unless the exponent of a Sylow 3-subgroup of G is 3.)

So we may assume that there is an automorphism of order 3 fixing a vertex and permuting its three neighbours transitively. Since G is vertex-transitive, it is arc-transitive.

Let v be any vertex, and N a minimal normal subgroup of G (which we may assume is an elementary abelian 2-group). We separate three cases, according to the behaviour of the neighbours of v.

Case 1: The neighbours of v are in the same N-orbit as v. In this case, N is transitive, so G is a Cayley graph, which is dealt with by the same argument as before.

Case 2: The neighbours of v are all in a single N-orbit which doesn't contain v. In this case, as before, there are just two N-orbits and G is bipartite; we find easily that it is the 3-cube.

Case 3: The neighbours of v are all in different N-orbits. In this case, the edges between two orbits (if any) form a 1-factor; the graph obtained by shrinking each N-orbit to a single vertex and each such 1-factor to a single edge is a cubic vertex-transitive graph, so has a semiregular automorphism of order greater than 2, by induction. This lifts to a semiregular automorphism group of G which is not an elementary abelian 2-group, and hence contains an element of order greater than 2. $\square$

Remark In an earlier version, we asked whether the following stronger version of the Lemma is true:

> If P is a 2-group which is not elementary abelian, then some non-identity element of the centre of P is a square.

It was pointed out to us by Alexander Hulpke and Andreas Caranti that this is not the case: there are counterexamples of order 128, for example, group number 36 in the list of small groups in GAP [2].

We conclude with an example to show that, if our conjecture is true, the function $f(n)$ cannot grow faster than $n^{1/3}$.

Let p be a prime congruent to ± 1 mod 16, and let G be the group $\mathrm{PSL}(2,p)$. Then G has a maximal subgroup H isomorphic to S_4. This subgroup contains a dihedral subgroup K of order 8, with $N_G(K)$ a dihedral group of order 16. (See Burnside [1] for the subgroups of G.) Then G, acting on the cosets of H, preserves the orbital graph corresponding to the double coset HxH, where

$x \in N_G(K) \setminus K$. Since $H \cap x^{-1}Hx = K$, and $|H : K| = 3$, this orbital graph is cubic and 1-transitive. Since $|H| = 3 \cdot 2^3$, it is 4-transitive. Now it follows that G is the full automorphism group. For Tutte's Theorem [4] shows that the full automorphism group has at most twice the order of G. So it contains G as a normal subgroup of index at most 2. If it is larger than G, it would be $\mathrm{PGL}(2,p)$. But this group does not contain a subgroup isomorphic to $S_4 \times C_2$.

Now the largest order of an element of G is p, and the number of vertices of the graph is $(p^3 - p)/48$.

References

[1] W. Burnside, *Theory of Groups of Finite Order*, Dover Publications (reprint), New York, 1955.

[2] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.2; Aachen, St Andrews, 1999. (`http://www-gap.dcs.st-and.ac.uk/~gap`)

[3] D. Marušič and R. Scapellato, Permutation groups, vertex-transitive digraphs and semiregular automorphisms, *Europ. J. Combinatorics* **19** (1998), 707–712.

[4] W. T. Tutte, A family of cubical graphs, *Proc. Cambridge Philos. Soc.* **43** (1947), 459–474.