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CRESTED PRODUCTS OF ASSOCIATION SCHEMES

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Abstract

In this paper, we define a new type of product of association schemes (and of the related objects, permutation groups and orthogonal block structures), which generalizes the direct and wreath products (which are referred to as "crossing" and "nesting" in the statistical literature.) Given two association schemes $\mathcal{Q}_r$ for $r = 1, 2$, each having an inherent partition F_r (that is, a partition whose equivalence relation is a union of adjacency relations in the association scheme), we define a product of the two schemes, which reduces to the direct product if $F_1 = U_1$ or $F_2 = E_2$, and to the wreath product if $F_1 = E_1$ and $F_2 = U_2$, where E_r and U_r are the relation of equality and the universal relation on $\mathcal{Q}_r$. We calculate the character table of the crested product, and show that if the two schemes $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have formal duals, then so does their crested product (and we give a simple description of this dual). We make an analogous definition for permutation groups with intransitive normal subgroups, and show that the constructions for association schemes and permutation groups are related in a natural way.

The definition can be generalized to association schemes with families of inherent partitions, or permutation groups with families of intransitive normal subgroups. This time the correspondence is not so straightforward, and works as expected only if the inherent partitions (or orbit partitions) form a distributive lattice.

We conclude with some open problems.

1. *Introduction*

Following Bose and Shimamoto [**8**], we define an *association scheme* on a finite set Ω to be a partition of $\Omega \times \Omega$ into classes $\mathcal{C}_i$, for i in $\mathcal{K}$, whose $(0, 1)$ adjacency matrices A_i are symmetric and satisfy

(i) there is a distinguished element 0 in $\mathcal{K}$ such that $A_0 = I_\Omega$, the identity matrix on Ω;

(ii) for all i, j in $\mathcal{K}$, the product A_iA_j is an integer linear combination of the A_k for k in $\mathcal{K}$.

It follows that $\sum_{i\in\mathcal{K}} A_i = J_\Omega$, the all-1 matrix on Ω, and that the span $\mathcal{A}$ of $\{A_i : i \in \mathcal{K}\}$ over $\mathbb{R}$ is a commutative algebra, called the *Bose–Mesner* algebra of the scheme. The *rank* of the scheme is $|\mathcal{K}|$. Note that we do not follow Bannai [**6**] in allowing non-symmetric adjacency matrices, apart from a remark at the end of Section 7.

The *trivial* association scheme on Ω has $A_0 = I_\Omega$ and $A_1 = J_\Omega - I_\Omega$. The isomorphism type of this association scheme is denoted by $\underline{n}$, where $n = |\Omega|$.

Given two association schemes $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on sets Ω_1 and Ω_2, with adjacency matrices A_i $(i \in \mathcal{K}_1)$ and B_j $(j \in \mathcal{K}_2)$, there are two well-established methods of combining them into a product association scheme on $\Omega_1 \times \Omega_2$. The two methods were formalized by Nelder in a statistical context [**15**]. One is called *crossing*: it yields the *direct product* $\mathcal{Q}_1 \times \mathcal{Q}_2$, whose adjacency matrices are

$$A_i \otimes B_j \qquad \text{for } i \text{ in } \mathcal{K}_1 \text{ and } j \text{ in } \mathcal{K}_2.$$

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The other is called *nesting*: it yields the *wreath product* $\mathcal{Q}_1/\mathcal{Q}_2$ (also denoted $\mathcal{Q}_2 \wr \mathcal{Q}_1$), whose adjacency matrices are

$$A_i \otimes J_{\Omega_2} \qquad \text{for } i \text{ in } \mathcal{K}_1 \setminus \{0\}$$

and

$$I_{\Omega_1} \otimes B_j \qquad \text{for } j \text{ in } \mathcal{K}_2.$$

For example, $\underline{\underline{n}} \times \underline{\underline{m}}$ is the rectangular association scheme $\mathrm{R}(n,m)$; while $\underline{\underline{n}}/\underline{\underline{m}}$ is the group-divisible scheme $\mathrm{GD}(n,m)$ with n blocks of size m.

The purpose of this paper is to give a single method of combining $\mathcal{Q}_1$ and $\mathcal{Q}_2$ into an association scheme on $\Omega_1 \times \Omega_2$ that has both crossing and nesting as special cases. The new method can be described very naturally for the class of association schemes derived from orthogonal block structures. This is done in Section 2, where necessary information about orthogonal block structures is summarized. Orthogonal block structures are defined by partitions. Section 3 gives some results about partitions in general association schemes. These are used in Section 4 to extend the new product to general association schemes, using a distinguished partition in each scheme. Section 5 gives the character table of the crested product of two schemes, and Section 6 uses this to explore duality of crested products.

Direct and wreath products are also established ways of combining two permutation groups. Section 7 gives the crested product of permutation groups, in such a way that if each group preserves an association scheme then the crested product of the groups preserves the crested product of the schemes.

Finally, a more general version of the crested product is presented in Sections 8–10. Here a whole family of partitions is needed in each scheme, rather than just one. Pointers to further work are in Section 11.

2. *Orthogonal block structures*

An orthogonal block structure on a set Ω is a set of partitions of Ω satisfying some conditions. Orthogonal block structures were introduced in [**17**]. The following summary is based on [**2**].

Given a partition F of Ω, denote by R_F the $\Omega \times \Omega$ relation matrix for F; that is $R_F(\alpha,\beta) = 1$ if α and β are in the same part of F, while $R_F(\alpha,\beta) = 0$ otherwise. A partition is *uniform* if all its parts have the same size; if F is uniform its part size is denoted by k_F. The two trivial partitions are the universal partition U, which has a single part, and the equality partition E, all of whose parts are singletons.

Partitions of Ω are partially ordered by the relation $\preccurlyeq$, where $F \preccurlyeq G$ if every part of F is contained in a part of G. Given any two partitions F and G, their *infimum* is the partition $F \wedge G$ whose parts are intersections of F-parts with G-parts. It is the coarsest partition which is finer than both F and G. Their *supremum* is the partition $F \vee G$ whose parts are minimal subject to being unions of F-parts and unions of G-parts.

Definition. A set $\mathcal{F}$ of uniform partitions of Ω is an *orthogonal block structure* if

(i) $\mathcal{F}$ contains U and E;
(ii) for all F and G in $\mathcal{F}$, $\mathcal{F}$ contains $F \wedge G$ and $F \vee G$;
(iii) for all F and G in $\mathcal{F}$, the matrices R_F and R_G commute with each other.

Given a partition F in an orthogonal block structure $\mathcal{F}$ on Ω, define the adjacency matrix A_F by

$$A_F(\alpha,\beta) = \begin{cases} 1 & \text{if } F = \bigwedge\{G \in \mathcal{F} : R_G(\alpha,\beta) = 1\} \\ 0 & \text{otherwise.} \end{cases}$$

It is shown in [**2**, **17**] that $\{A_F : F \in \mathcal{F},\ A_F \neq 0\}$ is an association scheme on Ω.

Now let F and G be partitions of sets Ω_1 and Ω_2 respectively. Define $F \times G$ to be the partition of $\Omega_1 \times \Omega_2$ whose relation matrix is $R_F \otimes R_G$. If F and G are both uniform then so is $F \times G$, and $k_{F\times G} = k_F \times k_G$.

Given orthogonal block structures $\mathcal{F}$ and $\mathcal{G}$ on Ω_1 and Ω_2 respectively, we can cross them to obtain $\mathcal{F} \times \mathcal{G}$ or nest them to obtain $\mathcal{F}/\mathcal{G}$. Here

$$\mathcal{F} \times \mathcal{G} = \{F \times G : F \in \mathcal{F},\ G \in \mathcal{G}\}$$

while

$$\mathcal{F}/\mathcal{G} = \{F \times U_2 : F \in \mathcal{F}\} \cup \{E_1 \times G : G \in \mathcal{G}\},$$

where E_r and U_r are the trivial partitions of Ω_r. It is shown in [**17**] that $\mathcal{F} \times \mathcal{G}$ and $\mathcal{F}/\mathcal{G}$ are both orthogonal block structures on $\Omega_1 \times \Omega_2$. Furthermore, the operation of deriving the association scheme from the orthogonal block structure commutes with both crossing and nesting. Thus the notation $\underline{\underline{n}}$ can be used unambiguously for both the trivial association scheme on an n-set and the trivial orthogonal block structure $\{E, U\}$ on an n-set, while $\underline{\underline{n}}/\underline{\underline{m}}$ denotes both the group-divisible scheme with n blocks of size m and also n copies of the trivial orthogonal block structure on an m-set.

We can now define the new way of combining two orthogonal block structures.

Definition. For $r = 1, 2$, let $\mathcal{F}_r$ be an orthogonal block structure on a set Ω_r and let $F_r \in \mathcal{F}_r$. The (simple) *crested product* of $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to F_1 and F_2 is the following set $\mathcal{G}$ of partitions of $\Omega_1 \times \Omega_2$:

$$\mathcal{G} = \{G_1 \times G_2 : G_1 \in \mathcal{F}_1,\ G_2 \in \mathcal{F}_2,\ G_1 \preccurlyeq F_1 \text{ or } G_2 \succcurlyeq F_2\}.$$

Theorem 1. *The crested product, as just defined, is an orthogonal block structure on* $\Omega_1 \times \Omega_2$.

Proof. All partitions in $\mathcal{G}$ are uniform, and all their relation matrices commute with each other. The two trivial partitions $U_1 \times U_2$ and $E_1 \times E_2$ of $\Omega_1 \times \Omega_2$ are in $\mathcal{G}$, because $U_2 \succcurlyeq F_2$ and $E_1 \preccurlyeq F_1$. Suppose that $G_1 \times G_2$ and $H_1 \times H_2$ are both in $\mathcal{G}$. Then $(G_1 \times G_2) \wedge (H_1 \times H_2) = (G_1 \wedge H_1) \times (G_2 \wedge H_2)$ and $(G_1 \times G_2) \vee (H_1 \times H_2) = (G_1 \vee H_1) \times (G_2 \vee H_2)$. If $G_1 \preccurlyeq F_1$ or $H_1 \preccurlyeq F_1$ then $G_1 \wedge H_1 \preccurlyeq F_1$ and so $(G_1 \times G_2) \wedge (H_1 \times H_2) \in \mathcal{G}$; otherwise, $G_2 \succcurlyeq F_2$ and $H_2 \succcurlyeq F_2$ so $G_2 \wedge H_2 \succcurlyeq F_2$ and so $(G_1 \times G_2) \wedge (H_1 \times H_2) \in \mathcal{G}$. Similarly, $(G_1 \times G_2) \vee (H_1 \times H_2) \in \mathcal{G}$. □

If $F_1 = U_1$ or $F_2 = E_2$ then $\mathcal{G}$ is $\mathcal{F}_1 \times \mathcal{F}_2$; if $F_1 = E_1$ and $F_2 = U_2$ then $\mathcal{G} = \mathcal{F}_1/\mathcal{F}_2$. Thus both crossing and nesting are special cases of the crested product. The word "crested" is a mixture of "crossed" and "nested" and is also cognate with the meaning of "wreath" in "wreath product".

Example 1. Any Latin square of order n defines an orthogonal block structure on the set of n^2 cells of the square: the nontrivial partitions R, C and L have as their parts the rows, columns and letters respectively. Take $\mathcal{F}_1$ to be such an orthogonal block structure and $\mathcal{F}_2$ to be trivial. The crested product of $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to L and U_2 has the Hasse diagram shown in Figure 1.

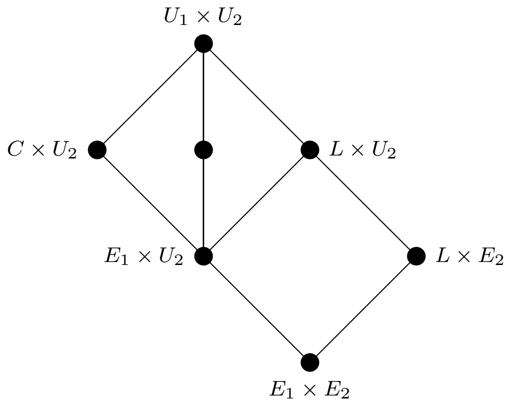


FIGURE 1. *Crested product of a Latin square and a trivial orthogonal block structure*

An important subclass of orthogonal block structures consists of the poset block structures. Suppose that $\leqslant$ is a partial order on a finite set X. A subset Y of X is defined to be *ancestral* if $y \in Y$ whenever $x \in Y$ and $x \leqslant y$. Denote by $\mathcal{S}(X)$ the set of ancestral subsets of X. For x in X, let Γ_x be a set of finite cardinality greater than 1, and put $\Omega = \prod_{x\in X} \Gamma_x$. Each subset Y of X defines a partition $F(Y)$ of Ω as follows: $(\gamma_x : x \in X)$ and $(\delta_x : x \in X)$ are in the same part of $F(Y)$ if and only if $\gamma_x = \delta_x$ for all x in Y. Note that $F(Y) \preccurlyeq F(Z)$ if and only if $Z \subseteq Y$. It is shown in [**2**] that $\{F(Y) : Y \in \mathcal{S}(X)\}$ is an orthogonal block structure on Ω. It is called a *poset block structure.*

THEOREM 2. *Crested products of poset block structures are poset block structures.*

Proof. For $r = 1, 2$, let $\leqslant_r$ be a partial order on X_r. Assume that $X_1 \cap X_2 = \emptyset$, and put $X = X_1 \cup X_2$. Let $\Omega_1 = \prod_{x\in X_1} \Gamma_x$ and $\Omega_2 = \prod_{x\in X_2} \Gamma_x$, where each $|\Gamma_x|$ is finite and at least 2. Then $\Omega_1 \times \Omega_2 = \prod_{x\in X} \Gamma_x$. For $r = 1, 2$, let $\mathcal{P}_r$ be the poset block structure on Ω_r defined by $(X_r, \leqslant_r)$, and let Y_r be an ancestral subset of X_r. If $\mathcal{P}$ is the crested product of $\mathcal{P}_1$ and $\mathcal{P}_2$ with respect to $F(Y_1)$ and $F(Y_2)$ then $\mathcal{P} = \{F(Z) : Z \in \mathcal{T}\}$ where

$$\mathcal{T} = \{Z_1 \cup Z_2 : Z_1 \in \mathcal{S}(X_1),\ Z_2 \in \mathcal{S}(X_2),\ Z_1 \supseteq Y_1 \text{ or } Z_2 \subseteq Y_2\}.$$

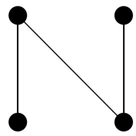
The elements of $\mathcal{T}$ are precisely those subsets of X which are ancestral for the partial order $\leqslant$ defined by

$$x \leqslant y \quad \text{if} \quad \begin{cases} x \in X_1,\ y \in X_1 \text{ and } x \leqslant_1 y, \text{ or} \\ x \in X_2,\ y \in X_2 \text{ and } x \leqslant_2 y, \text{ or} \\ x \in X_2 \setminus Y_2 \text{ and } y \in Y_1. \end{cases} \tag{2.1}$$

Hence $\mathcal{P}$ is the poset block structure on $\Omega_1 \times \Omega_2$ defined by $(X, \leqslant)$. $\square$

If $Y_1 = \emptyset$ or $Y_2 = X_2$ then $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ and $(X, \leqslant)$ is the *cardinal sum* of the posets $(X_1, \leqslant_1)$ and $(X_2, \leqslant_2)$; if $Y_1 = X_1$ and $Y_2 = \emptyset$ then $\mathcal{P} = \mathcal{P}_1/\mathcal{P}_2$ and $(X, \leqslant)$ is the *ordinal sum* of the posets $(X_1, \leqslant_1)$ and $(X_2, \leqslant_2)$. Thus Equation (2.1) is a way of combining two disjoint posets that generalizes both cardinal sum and ordinal sum.

When only crossing and nesting are available as binary operators on orthogonal block structures, not all poset block structures can be built up from trivial ones. For example, the poset



cannot be obtained from singleton posets by repeated use of cardinal and/or ordinal sum. Crested products change this completely.

Theorem 3. *Every poset block structure can be obtained from trivial poset block structures by repeated use of crested products.*

Proof. It suffices to prove that every finite poset can be built from two disjoint smaller posets by means of Construction (2.1).

Let $(X, \leqslant)$ be a finite poset and let y be a maximal element of X. Put $X_1 = Y_1 = \{y\}$, $X_2 = X \setminus \{y\}$ and $Y_2 = X_2 \setminus \{x \in X_2 : x < y\}$. Let $\leqslant_1$ be the trivial partial order on X_1 and let $\leqslant_2$ be the restriction of $\leqslant$ to X_2. Then $Y_1 \in \mathcal{S}(X_1)$, $Y_2 \in \mathcal{S}(X_2)$ and $\leqslant$ is obtained from $\leqslant_1$ and $\leqslant_2$ by Construction (2.1). □

Theorem 3 does not extend to orthogonal block structures. The lattice of partitions in an orthogonal block structure is modular, so the lattice has a well-defined height. Since the crested product of $\mathcal{F}_1$ and $\mathcal{F}_2$ always contains

$$F \times U_2 \qquad \text{for all } F \text{ in } \mathcal{F}_1$$

and

$$E_1 \times G \qquad \text{for all } G \text{ in } \mathcal{F}_2,$$

the height of the crested product is the sum of the heights of $\mathcal{F}_1$ and $\mathcal{F}_2$.

It follows that the only crested products of height two are $\underline{\underline{n}}/\underline{\underline{m}}$ and $\underline{\underline{n}} \times \underline{\underline{m}}$. The orthogonal block structure defined by a Latin square in Example 1 has height two and cannot be obtained as a crested product. Nor can the other orthogonal block structures of height two: these are obtained from sets of mutually orthogonal Latin squares, with $t + 2$ non-trivial partitions if there are t squares.

Example 2. Similar arguments show that the orthogonal block structure shown in Figure 2 is not a crested product. Here Ω is the group

$$\langle a, b : a^9 = b^3 = 1,\ b^{-1}ab = a^4 \rangle.$$

Each subgroup of Ω defines a uniform partition of Ω into its left cosets. In this group all subgroups commute in pairs, so these partitions do form an orthogonal block structure, which has height three. Examination of all possible crested products of orthogonal block structures with heights one and two shows that none of them is the one in Figure 2.

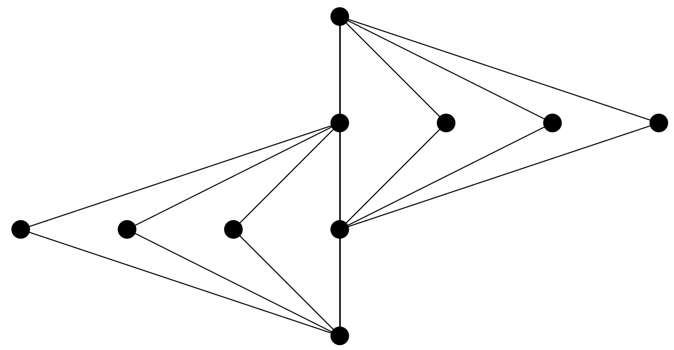


FIGURE 2. *Orthogonal block structure defined by the subgroups of the group in Example 2*

3. *Partitions in association schemes*

DEFINITION. Let $\mathcal{Q}$ be an association scheme on Ω with adjacency matrices A_i for i in $\mathcal{K}$. A partition F of Ω is *inherent* in $\mathcal{Q}$ if there is a subset $\mathcal{L}$ of $\mathcal{K}$ such that $R_F = \sum_{i\in\mathcal{L}} A_i$.

The trivial partitions E and U are inherent in every association scheme. If there are no other inherent partitions then the association scheme is called *primitive*; otherwise it is imprimitive: see [**11**]. Thus all non-trivial orthogonal block structures are imprimitive.

EXAMPLE 3. There is an association scheme on the 12 edges of the cube. Distinct edges α and β are related by relation

1	if α and β meet at a vertex
2	if α and β are diagonally opposite
3	if α and β are parallel but not opposite
4	if α and β are skew.

The partitions inherent in the association scheme have relation matrices A_0, A_0+A_2, $A_0 + A_2 + A_3$ and $A_0 + A_1 + A_2 + A_3 + A_4$.

THEOREM 4. *If $\mathcal{Q}$ is an association scheme on Ω then the set $\mathcal{F}$ of partitions of Ω which are inherent in $\mathcal{Q}$ is an orthogonal block structure on Ω.*

Proof. Every adjacency matrix of $\mathcal{Q}$ has constant row-sums, so every partition in $\mathcal{F}$ is uniform. Moreover, $\mathcal{F}$ contains U and E.

Suppose that F and G are in $\mathcal{F}$. Then R_F and R_G are in the Bose–Mesner algebra $\mathcal{A}$ of $\mathcal{Q}$, which is commutative, so R_F commutes with R_G. If $R_F = \sum_{i\in\mathcal{L}} A_i$ and $R_G = \sum_{i\in\mathcal{M}} A_i$ then $R_{F\wedge G} = \sum_{i\in\mathcal{L}\cap\mathcal{M}} A_i$, and so $F\wedge G$ is in $\mathcal{F}$. In particular, $F\wedge G$ is uniform, and so Proposition 3 of [**2**] shows that $R_F R_G = k_{F\wedge G} R_{F\vee G}$: therefore $R_{F\vee G}\in\mathcal{A}$ and so $F\vee G\in\mathcal{F}$. □

Theorem 4 gives a more direct proof of Theorem 7 of [**3**].

EXAMPLE 4. Let Ω be an Abelian group and let $\mathcal{Q}$ be the association scheme on Ω in which (α,β) is in the same class as (γ,δ) if $\alpha^{-1}\beta \in \left\{\gamma^{-1}\delta, \delta^{-1}\gamma\right\}$. The partitions inherent in $\mathcal{Q}$ form the orthogonal block structure defined by all subgroups of Ω: see [**1**, **9**].

Given any partition $\mathcal{P}$ of $\Omega\times\Omega$, let $V(\mathcal{P})$ be the span (over $\mathbb{R}$) of the adjacency matrices of its classes. Then $\mathcal{Q}\preccurlyeq\mathcal{P}$ if and only if $V(\mathcal{P})\leqslant\mathcal{A}$.

DEFINITION. Let $\mathcal{Q}$ be an association scheme on Ω. A partition $\mathcal{P}$ of $\Omega\times\Omega$ is *ideal* for $\mathcal{Q}$ if $V(\mathcal{P})$ is an ideal of $\mathcal{A}$ in the sense that $V(\mathcal{P})\leqslant\mathcal{A}$ and $AD\in V(\mathcal{P})$ whenever $A\in\mathcal{A}$ and $D\in V(\mathcal{P})$.

Inherent partitions were introduced in [**11**] in order to define quotient schemes. The calculations in the proof of Theorem 9.4 of [**7**] show that if F is an inherent partition of $\mathcal{Q}$ then there is an ideal partition $\vartheta(F)$ for $\mathcal{Q}$ such that A_iR_F is an integer multiple of an adjacency matrix of $\vartheta(F)$ for all i in $\mathcal{K}$. The following result shows that ϑ is a bijection.

THEOREM 5. *Let $\mathcal{P}$ be an ideal partition for $\mathcal{Q}$. Let the adjacency matrices for $\mathcal{Q}$ be A_i for i in $\mathcal{K}$, and those for $\mathcal{P}$ be D_m for m in $\mathcal{M}$. Denote by σ the surjection from $\mathcal{K}$ to $\mathcal{M}$ such that class i of $\mathcal{Q}$ is contained in class $\sigma(i)$ of $\mathcal{P}$. Put $R = D_{\sigma(0)}$. Then R is the relation matrix of an inherent partition in $\mathcal{Q}$. Moreover, for all i in $\mathcal{K}$, A_iR is an integer multiple of $D_{\sigma(i)}$.*

Proof. First fix i in $\mathcal{K}$. Because $\mathcal{P}$ is an ideal partition, there are integers t_m, for m in $\mathcal{M}$, such that $A_iR = \sum_{m\in\mathcal{M}} t_m D_m$. If m and n are distinct elements of $\mathcal{M}$ then the diagonal elements of D_mD_n are zero; while those of D_m^2 are equal to the constant row-sum d_m of D_m. Therefore the diagonal elements of A_iRD_m are equal to t_md_m. If $m\neq\sigma(i)$ then the diagonal elements of A_iD_m are zero, but $A_iD_m\in V(\mathcal{P})$ so there are integers u_l, for l in $\mathcal{M}\setminus\{\sigma(0)\}$, such that

$$A_iD_m = \sum_{l\in\mathcal{M}\setminus\{\sigma(0)\}} u_lD_l.$$

Then $A_iRD_m = A_iD_mR = A_iD_mD_{\sigma(0)} = \sum_{l\neq\sigma(0)} u_lD_lD_{\sigma(0)}$, whose diagonal elements are zero, so $t_m = 0$. Hence $A_iR = t_{\sigma(i)}D_{\sigma(i)}$.

Now put $\mathcal{L} = \{i\in\mathcal{K} : \sigma(i) = \sigma(0)\}$. Then $R = \sum_{i\in\mathcal{L}} A_i$ and A_iR is an integer multiple of R for all i in $\mathcal{L}$. Consequently R^2 is an integer multiple of R and so R is the relation matrix of a uniform partition of Ω. This partition is inherent because R is in the Bose–Mesner algebra of $\mathcal{Q}$. □

In the quotient association scheme of $\mathcal{Q}$ by its inherent partition F, the objects are the parts of F and the adjacency matrices are the collapsed versions of the adjacency matrices of $\vartheta(F)$.

4. *Products of association schemes*

Let F be a partition in an orthogonal block structure $\mathcal{F}$. Then $R_F = \sum_{G\in\mathcal{L}} A_G$, where $\mathcal{L} = \{G \in \mathcal{F} : G \preccurlyeq F\}$. Hence F is inherent in the association scheme $\mathcal{Q}$ derived from $\mathcal{F}$. Let $\mathcal{A}$ be the Bose–Mesner algebra of $\mathcal{Q}$. Then $\{A_G : G \in \mathcal{L}\}$ and $\{R_G : G \in \mathcal{L}\}$ span the same subspace $\mathcal{A}|_F$ of $\mathcal{A}$, and this is closed under matrix multiplication.

Let $\mathcal{P}$ be the ideal partition $\vartheta(F)$. For G in $\mathcal{F}$, R_G is in the ideal of $\mathcal{A}$ generated by R_F if and only if $F \preccurlyeq G$: therefore $V(\mathcal{P})$ is the span of $\{R_G : G \in \mathcal{F},\ G \succcurlyeq F\}$. Henceforth write $\mathcal{A}|^F$ for $V(\vartheta(F))$.

Now let $\mathcal{G}$ be the crested product of orthogonal block structures $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to F_1 and F_2. The span of the relation matrices of the partitions in $\mathcal{G}$ is

$$(\mathcal{A}_1|_{F_1} \otimes \mathcal{A}_2) + (\mathcal{A}_1 \otimes \mathcal{A}_2|^{F_2}),$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the Bose–Mesner algebras of the association schemes defined by $\mathcal{F}_1$ and $\mathcal{F}_2$. The adjacency matrices of the association scheme defined by $\mathcal{G}$ are $(0,1)$-matrices which sum to $J_{\Omega_1\times\Omega_2}$ and which span this algebra. Therefore they are

$$A_G \otimes A_H \qquad \text{for } G \text{ in } \mathcal{L} \text{ and } H \text{ in } \mathcal{F}_2$$

and

$$A_G \otimes D \qquad \text{for } G \text{ in } \mathcal{F}_1 \setminus \mathcal{L} \text{ and } D \text{ an adjacency matrix of } \mathcal{P},$$

where $\mathcal{L} = \{G \in \mathcal{F}_1 : G \preccurlyeq F_1\}$ and $\mathcal{P} = \vartheta(F_2)$. This motivates the following definition.

Definition. For $r = 1, 2$, let $\mathcal{Q}_r$ be an association scheme on a set Ω_r, and let F_r be an inherent partition in $\mathcal{Q}_r$. Put $\mathcal{P} = \vartheta(F_2)$ and $\Omega = \Omega_1 \times \Omega_2$. Let the adjacency matrices of $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{P}$ be A_i for i in $\mathcal{K}_1$, B_j for j in $\mathcal{K}_2$, and D_m for m in $\mathcal{M}$. Let $\mathcal{L}$ be the subset of $\mathcal{K}_1$ such that $R_{F_1} = \sum_{i\in\mathcal{L}} A_i$. The (simple) *crested product* of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with respect to F_1 and F_2 is the set $\mathcal{Q}$ of subsets of $\Omega \times \Omega$ whose adjacency matrices are

$$A_i \otimes B_j \qquad \text{for } i \text{ in } \mathcal{L} \text{ and } j \text{ in } \mathcal{K}_2$$

and

$$A_i \otimes D_m \qquad \text{for } i \text{ in } \mathcal{K}_1 \setminus \mathcal{L} \text{ and } m \text{ in } \mathcal{M}.$$

Theorem 6. *The crested product, as just defined, is an association scheme on* Ω.

Proof. All of the matrices are symmetric. There are distinguished elements 0 in both $\mathcal{L}$ and $\mathcal{K}_2$ such that $A_0 = I_{\Omega_1}$ and $B_0 = I_{\Omega_2}$: then $A_0 \otimes B_0 = I_\Omega$.

We have

$$\sum_{i\in\mathcal{L}}\sum_{j\in\mathcal{K}_2}(A_i \otimes B_j) = \left(\sum_{i\in\mathcal{L}} A_i\right) \otimes \left(\sum_{j\in\mathcal{K}_2} B_j\right) = R_{F_1} \otimes J_{\Omega_2}$$

and

$$\sum_{i\in\mathcal{K}_1\setminus\mathcal{L}}\sum_{m\in\mathcal{M}}(A_i\otimes D_m)=\left(\sum_{i\in\mathcal{K}_1\setminus\mathcal{L}}A_i\right)\otimes\left(\sum_{m\in\mathcal{M}}D_m\right)=(J_{\Omega_1}-R_{F_1})\otimes J_{\Omega_2}$$

so the subsets of $\Omega\times\Omega$ defined by the adjacency matrices do form a partition of $\Omega\times\Omega$.

For $r=1$, 2, let $\mathcal{A}_r$ be the Bose–Mesner algebra of $\mathcal{Q}_r$. Let $\mathcal{A}_1|_{F_1}$ be the span of $\{A_i : i\in\mathcal{L}\}$. Then $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_1|_{F_1}$ and $\mathcal{A}_2|^{F_2}$ are all closed under multiplication. Define $\sigma\colon\mathcal{K}_2\to\mathcal{M}$ as in Theorem 5. Then

$$\sum_{j:\sigma(j)=m}A_i\otimes B_j=A_i\otimes D_m$$

and so the span $\mathcal{A}$ of the adjacency matrices is $(\mathcal{A}_1|_{F_1}\otimes\mathcal{A}_2)+(\mathcal{A}_1\otimes\mathcal{A}_2|^{F_2})$. Each of $(\mathcal{A}_1|_{F_1}\otimes\mathcal{A}_2)$ and $(\mathcal{A}_1\otimes\mathcal{A}_2|^{F_2})$ is closed under matrix multiplication. If $i_1\in\mathcal{L}$, $j\in\mathcal{K}_2$, $i_2\in\mathcal{K}_1$ and $m\in\mathcal{M}$ then $A_{i_1}A_{i_2}\in\mathcal{A}_1$ and $B_jD_m\in\mathcal{A}_2|^{F_2}$, because $\mathcal{P}$ is ideal, and so $(A_{i_1}\otimes B_j)(A_{i_2}\otimes D_m)\in\mathcal{A}$. Hence $\mathcal{A}$ is closed under multiplication and so $\mathcal{Q}$ is an association scheme. □

Again, the crested product reduces to the direct product if $F_1=U_1$ or $F_2=E_2$ (in which case $\mathcal{P}=\mathcal{Q}_2$); and it reduces to the wreath product if $F_1=E_1$ and $F_2=U_2$ (in which case $\mathcal{P}=U_{\Omega_2\times\Omega_2}$). The crested product is always imprimitive, because it contains the inherent partition $E_1\times U_2$. The subscheme induced on each class of $E_1\times U_2$ is isomorphic to $\mathcal{Q}_2$, while the quotient scheme is isomorphic to $\mathcal{Q}_1$. This gives a flexible method of constructing an 'extension' of one association scheme by another.

5. *The character table of a crested product*

Since the Bose–Mesner algebra $\mathcal{A}$ of an association scheme $\mathcal{Q}$ is commutative, it has common eigenspaces, called *strata* by statisticians. The stratum projectors S_e, for e in $\mathcal{E}$, are also known as *primitive idempotents*. They form an alternative basis for $\mathcal{A}$, so $|\mathcal{K}|=|\mathcal{E}|$: see [**12**, Chapter 17]. The *character table* of $\mathcal{Q}$ is the $\mathcal{K}\times\mathcal{E}$ matrix whose entry $C(i,e)$ is the eigenvalue of A_i on stratum W_e (this matrix is called P in [**12**]).

If F is an inherent partition in $\mathcal{Q}$ then $k_F^{-1}R_F$ is idempotent so there is a subset $\mathcal{H}$ of $\mathcal{E}$ such that $R_F=k_F\sum_{e\in\mathcal{H}}S_e$. Define the equivalence relation $\sim$ on $\mathcal{E}$ by $e\sim f$ if and only if $C(i,e)=C(i,f)$ for all i in $\mathcal{L}$, and write $[e]$ for the equivalence class containing e. Then $\mathcal{H}=[0]$. The classes of each of the subschemes defined by F are indexed by $\mathcal{L}$ while the strata of each are indexed by the equivalence classes of $\sim$. The Bose–Mesner algebra of each subscheme is isomorphic to $\mathcal{A}|_F$.

On the other hand, the classes of the quotient scheme are indexed by the classes of $\vartheta(F)$ while the stratum projectors are collapsed versions of S_e for e in $\mathcal{H}$. The linear map τ that takes $k_F^{-1}D_m$ to the m-th adjacency matrix of the quotient scheme is an isomorphism from $\mathcal{A}|^F$ to the Bose–Mesner algebra of the quotient scheme.

THEOREM 7. *Let $\mathcal{Q}$ be the crested product defined in Section 4. Let the strata for $\mathcal{Q}_1$ be W_e, for e in $\mathcal{E}_1$, with stratum projectors S_e, and those for $\mathcal{Q}_2$ be V_f, for f in $\mathcal{E}_2$, with stratum projectors T_f. For $r=1$, 2, let C_r be the character table of $\mathcal{Q}_r$.*

In $\mathcal{Q}_1$, put $e \sim f$ if $C_1(i,e) = C_1(i,f)$ for all i in $\mathcal{L}$, and put $U_{[e]} = \bigoplus_{f\sim e} W_f$, whose projector is equal to $\sum_{f\sim e} S_f$. In $\mathcal{Q}_2$, let $\mathcal{H}$ be the subset of $\mathcal{E}_2$ such that $R_{F_2} = k_{F_2}\sum_{f\in\mathcal{H}} T_f$. Then the strata for $\mathcal{Q}$ are $W_e \otimes V_f$, for (e,f) in $\mathcal{E}_1 \times \mathcal{H}$, and $U_{[e]} \otimes V_f$, for equivalence classes $[e]$ of $\sim$ and f in $\mathcal{E}_2 \setminus \mathcal{H}$. The eigenvalues are as follows, where $\bar{C}_2$ is the character table of the quotient $\bar{\mathcal{Q}}_2$ of $\mathcal{Q}_2$ by F_2.

	$W_e \otimes V_f \quad (f \in \mathcal{H})$	$U_{[e]} \otimes V_f \quad (f \notin \mathcal{H})$
$A_i \otimes D_m \quad (i \notin \mathcal{L})$	$C_1(i,e)k_{F_2}\bar{C}_2(m,f)$	0
$A_i \otimes B_j \quad (i \in \mathcal{L})$	$C_1(i,e)C_2(j,f)$	$C_1(i,e)C_2(j,f)$

Proof. The claimed strata are mutually orthogonal and sum to $\mathbb{R}^{\Omega_1\times\Omega_2}$.

If $f \notin \mathcal{H}$ then $R_{F_2}T_f = 0$ and so $D_m T_f = 0$ because D_m is a multiple of R_{F_2}. The other eigenvalues given follow directly from the character tables of the two schemes, the definition of $\sim$ and the isomorphism τ. Thus to show that none of the claimed strata merge into a single stratum it is sufficient to show that we have the correct number of strata.

The number of equivalence classes of $\sim$ is equal to the rank of the subscheme of $\mathcal{Q}_1$ induced on each part of F_1, which is $|\mathcal{L}|$, while $|\mathcal{M}|$ and $|\mathcal{H}|$ are both equal to the rank of $\bar{\mathcal{Q}}_2$. Hence the number of adjacency matrices and the number of claimed strata are both equal to $|\mathcal{K}_1|\cdot|\mathcal{H}| - |\mathcal{L}|\cdot|\mathcal{H}| + |\mathcal{L}|\cdot|\mathcal{K}_2|$. □

6. *Dual association schemes*

Association schemes $\mathcal{Q}$ and $\mathcal{Q}^*$ on sets of the same size n are said to be *formally dual* [**12**, Chapter 17] if there is a bijection $*$ from the classes of $\mathcal{Q}$ to the strata of $\mathcal{Q}^*$ and from the strata of $\mathcal{Q}$ to the classes of $\mathcal{Q}^*$ such that

$$C^*(e^*, i^*) = nC^{-1}(e,i) \tag{6.1}$$

for all (i,e) in $\mathcal{K}\times\mathcal{E}$, where C and C^* are the character tables of $\mathcal{Q}$ and $\mathcal{Q}^*$ respectively. Note that Equation (17.14) of [**12**] shows that

$$C^{-1}(e,i) = C(i,e)r_e/(nv_i), \tag{6.2}$$

where r_e is the rank of S_e and v_i is the valency of $\mathcal{C}_i$.

Suppose that $\mathcal{Q}$ has such a dual, and that F is an inherent partition in $\mathcal{Q}$ with $R_F = \sum_{i\in\mathcal{L}} A_i = k_F \sum_{e\in\mathcal{H}} S_e$. Then

$$\sum_{i\in\mathcal{L}} A_i = k_F \sum_{e\in\mathcal{H}}\sum_{j\in\mathcal{K}} C^{-1}(e,j)A_j$$

so $\sum_{e\in\mathcal{H}} C^{-1}(e,j)$ is equal to $1/k_F$ if $j\in\mathcal{L}$ and to zero otherwise. Hence, in $\mathcal{Q}^*$,

$$\sum_{e\in\mathcal{H}} A_{e^*} = \sum_{e\in\mathcal{H}}\sum_{i\in\mathcal{K}} C^*(e^*,i^*)S_{i^*} = \sum_{i\in\mathcal{K}}\sum_{e\in\mathcal{H}} nC^{-1}(e,i)S_{i^*} = \frac{n}{k_F}\sum_{i\in\mathcal{L}} S_{i^*}.$$

Therefore $\sum_{e\in\mathcal{H}} A_{e^*}$ is the relation matrix of an inherent partition F^* in $\mathcal{Q}^*$ whose parts have size n/k_F. We may call F^* the inherent partition dual to F.

To proceed, we need to give more explicit results about the character table of a quotient scheme than we have used so far. The conclusion of Theorem 5 enables us to change the notation used in its proof so that $A_iR_F = t_iD_{\sigma(i)}$ for i in $\mathcal{K}$. Moreover, $D_{\sigma(i)} = \sum_{j\in\sigma^{-1}\sigma(i)} A_j$. Then $v_ik_F = t_id_{\sigma(i)}$ and $d_{\sigma(i)} = \sum_{j\in\sigma^{-1}\sigma(i)} v_j$.

LEMMA 8. *Let $\bar{C}$ be the character table of the quotient scheme $\bar{\mathcal{Q}}$ of $\mathcal{Q}$ by F. Then $\bar{C}(\sigma(i), e) = C(i, e)/t_i$ and $\bar{C}^{-1}(e, \sigma(i)) = k_FC^{-1}(e, i)$ for e in $\mathcal{H}$ and i in $\mathcal{K}$.*

Proof. The primitive idempotents of $\bar{\mathcal{Q}}$ are $\tau(S_e)$ for e in $\mathcal{H}$, and the adjacency matrices are the distinct $\tau(k_F^{-1}D_{\sigma(i)})$. Now

$$\begin{aligned}\tau(k_F^{-1}D_{\sigma(i)}) &= \tau(t_i^{-1}A_ik_F^{-1}R_F)\\ &= \tau(t_i^{-1}A_i)\tau(k_F^{-1}R_F)\\ &= \tau(t_i^{-1}\sum_{e\in\mathcal{E}} C(i,e)S_e) \times I\\ &= \sum_{e\in\mathcal{E}} \frac{C(i,e)}{t_i}\tau(S_e)\end{aligned}$$

so $\bar{C}(\sigma(i), e) = C(i, e)/t_i$.

The valency $v_{\sigma(i)}$ of $\tau(D_{\sigma(i)})$ is just v_i/t_i, so Equation (6.2) shows that if there are n points in the scheme $\mathcal{Q}$ then

$$\bar{C}^{-1}(e, \sigma(i)) = \frac{\bar{C}(\sigma(i), e)r_e}{(n/k_F)v_{\sigma(i)}} = \frac{C(i, e)k_Fr_e}{t_in(v_i/t_i)} = k_FC^{-1}(e, i). \qquad \square$$

The second part of this lemma shows that if $\sigma(i) = \sigma(j)$ then $C^{-1}(e, i) = C^{-1}(e, j)$ for all e in $\mathcal{H}$. Hence if Equation (6.1) holds then $C^*(e^*, i^*) = C^*(e^*, j^*)$ for all e in $\mathcal{H}$, so $i^* \sim j^*$. Conversely, if $i^* \sim j^*$ and Equation (6.1) holds then $A_i \sum_{e\in\mathcal{H}} S_e$ is a scalar multiple of $A_j \sum_{e\in\mathcal{H}} S_e$, so $\sigma(i) = \sigma(j)$.

Incidentally, this lemma also shows that if $\mathcal{Q}$ is formally dual to $\mathcal{Q}^*$ and F is inherent in $\mathcal{Q}$ then the quotient scheme of $\mathcal{Q}$ by F is formally dual to the subschemes of $\mathcal{Q}^*$ induced on each part of F^*.

THEOREM 9. *For $r = 1$, 2, suppose that F_r is an inherent partition in the association scheme $\mathcal{Q}_r$ on a set of size n_r, and that $\mathcal{Q}_r^*$ is formally dual to $\mathcal{Q}_r$. Then the crested product $\mathcal{Q}^*$ of $\mathcal{Q}_1^*$ and $\mathcal{Q}_2^*$ with respect to F_1^* and F_2^* is formally dual to the crested product $\mathcal{Q}$ of $\mathcal{Q}_2$ and $\mathcal{Q}_1$ with respect to F_2 and F_1.*

Proof. The adjacency matrices of $\mathcal{Q}$ are $A_i \otimes D_m$, for i in $\mathcal{K}_1 \setminus \mathcal{L}$ and m in $\sigma(\mathcal{K}_2)$, and $A_i \otimes B_j$, for i in $\mathcal{L}$ and j in $\mathcal{K}_2$, while the strata are $W_e \otimes V_f$ for e in $\mathcal{E}_1$ and f in $\mathcal{H}$, and $U_{[e]} \otimes V_f$ for equivalence classes of $\sim$ on $\mathcal{E}_1$ and f in $\mathcal{E}_2 \setminus \mathcal{H}$. The adjacency matrices of $\mathcal{Q}^*$ are labelled by $(f^*, \sigma(e^*))$ for f in $\mathcal{E}_2 \setminus \mathcal{H}$ and e in $\mathcal{E}_1$, and by (f^*, e^*) for f in $\mathcal{H}$ and e in $\mathcal{E}_1$, while its strata are labelled by (j^*, i^*) for j in $\mathcal{K}_2$ and i in $\mathcal{L}$, and by $([j^*], i^*)$ for j in $\mathcal{K}_2$ and i in $\mathcal{K}_1 \setminus \mathcal{L}_1$.

Let C and C^* be the character tables of $\mathcal{Q}$ and $\mathcal{Q}^*$. We use Theorem 7 to obtain C, then Equation (6.2) to obtain C^{-1}, and show that it satisfies Equation (6.1) by using Theorem 7 for $\mathcal{Q}^*$.

First, take i in $\mathcal{L}$, j in $\mathcal{K}_2$, e in $\mathcal{E}_1$ and f in $\mathcal{H}$. Then $C((i, j), (e, f)) = C_1(i, e)C_2(j, f)$. The valency of $A_j \otimes B_j$ is v_iv_j, so

$$C^{-1}((e, f), (i, j)) = \frac{C_1(i, e)C_2(j, f)r_er_f}{n_1n_2v_iv_j}$$

$$= C_1^{-1}(e,i)C_2^{-1}(f,j)$$
$$= \frac{C_1^*(e^*,i^*)C_2^*(f^*,j^*)}{n_1n_2}$$
$$= (n_1n_2)^{-1}C^*((f^*,e^*),(j^*,i^*).$$

Secondly, take i in $\mathcal{K}_1 \setminus \mathcal{L}$, j in $\mathcal{K}_2$, e in $\mathcal{E}_1$ and f in $\mathcal{H}$. Then

$$C((i,\sigma(j)),(e,f)) = C_1(i,e)k_{F_2}\bar{C}_2(\sigma(j),f) = C_1(i,e)k_{F_2}C_2(j,f)/t_j.$$

The valency of $A_i \otimes D_{\sigma(j)}$ is $v_iv_jk_{F_2}/t_j$, so

$$C^{-1}((e,f),(i,\sigma(j)) = \frac{C_1(i,e)k_{F_2}C_2(j,f)r_er_f}{n_1n_2v_iv_jk_{F_2}}$$
$$= C_1^{-1}(i,e)C_2^{-1}(j,f)$$
$$= \frac{C_1^*(e^*,i^*)C_2^*(f^*,j^*)}{n_1n_2}$$
$$= (n_1n_2)^{-1}C^*((f^*,e^*),(j^*,i^*)).$$

Thirdly, take i in $\mathcal{L}$, j in $\mathcal{K}_2$, e in $\mathcal{E}_1$ and f in $\mathcal{E}_2 \setminus \mathcal{H}$. Then $C((i,j),([e],f)) = C_1(i,e)C_2(j,f)$. Now the dimension of the stratum is $r_f \sum_{g\in[e]} r_g$. However,

$$\sum_{g\in[e]} r_g = \sum_{g\in[e]} v_{g^*} = \sum_{g^*\in\sigma^{-1}\sigma(e)} v_{g^*} = k_{F_1^*}v_{e^*}/t_{e^*} = k_{F_1^*}r_e/t_{e^*},$$

so

$$C^{-1}(([e],f),(i,j)) = \frac{C_1(i,e)C_2(j,f)k_{F_1^*}r_er_f}{n_1n_2v_iv_jt_{e^*}}$$
$$= \frac{C_1^{-1}(e,i)C_2^{-1}(f,j)k_{F_1^*}}{t_{e^*}}$$
$$= \frac{C_1^*(e^*,i^*)k_{F_1^*}C_2^*(f^*,j^*)}{n_1n_2t_{e^*}}$$
$$= \frac{C_2^*(f^*,j^*)k_{F_1^*}\bar{C}_1^*(e^*,i^*)}{n_1n_2}$$
$$= (n_1n_2)^{-1}C^*((f^*,\sigma(e^*)),(j^*,i^*)).$$

Finally, take i in $\mathcal{K}_1\setminus\mathcal{L}$, j in $\mathcal{K}_2$, e in $\mathcal{E}_1$ and f in $\mathcal{E}_2\setminus\mathcal{H}$. Then $C((i,\sigma(i)),([e],f)) = 0$ and so $C^{-1}(([e],f),(i,\sigma(j))) = 0$. However, $[e] = \sigma(e^*)$ and $[j^*] = \sigma(j)$, and $C^*((f^*,\sigma(e^*),([j^*],i^*)) = 0$. □

7. *Products of permutation groups*

In this section we define the crested product of two permutation groups. We write permutations on the right, in contrast to other functions. We also warn of another source of confusion. We use G^Ω to denote the set of functions from Ω to G, not (as often in permutation group theory) the permutation group induced on the set Ω by the group G.

For $r = 1, 2$, let G_r be a transitive permutation group on Ω_r. We recall the definition of the wreath product $G_2 \wr G_1$, which is a permutation group on $\Omega_1 \times \Omega_2$ generated by the following two subgroups:

(a) the *base group* $B = G_2^{\Omega_1}$, acting by the rule

$$(\alpha_1, \alpha_2)\phi = (\alpha_1, \alpha_2\phi(\alpha_1))$$

for $\phi \in G_2^{\Omega_1}$;

(b) the *top group* $T = G_1$, acting by the rule

$$(\alpha_1, \alpha_2)g = (\alpha_1 g, \alpha_2)$$

for $g \in G_1$.

Note that T normalizes B, so the two groups generate their product; and $B \cap T = 1$, so the product is semi-direct.

If we replace the base group B by its subgroup G_2, embedded diagonally (that is, as the group of constant functions), we obtain the *direct product* $G_1 \times G_2$.

Now suppose that, for $r = 1, 2$, we have a partition F_r of Ω_r which is invariant under G_r. We need to make a further assumption: suppose that there is a normal subgroup N of G_2 such that F_2 is the orbit partition of N. Now we may as well assume that N consists of all elements of G_2 which fix all parts of the partition F_2. We use F_r also to denote the set of parts of the partition F_r.

The base group B of the simple crested product is the group generated by N^{F_1} and G_2. Here N^{F_1} is embedded in $G_2^{\Omega_1}$ as the set of functions which are constant on the classes of F_1 and take values in N; and G_2 is embedded diagonally, as before. Clearly G_2 normalizes N^{F_1}, so their product is a group; and the intersection is N (embedded diagonally). The top group is G_1, which normalizes B; so the crested product is again a semi-direct product of B by G_1. Its order is $|N|^{|F_1|-1} \cdot |G_1| \cdot |G_2|$. Clearly it contains the direct product and is contained in the wreath product of G_2 and G_1. Moreover, as usual, we see that it is equal to the direct product if $F_2 = E_2$ or if $F_1 = U_1$, and is equal to the wreath product if $F_2 = U_2$ and $F_1 = E_1$.

There are two concepts of an automorphism of an association scheme. In this paper we shall use the strong sense: a permutation of the underlying set that preserves all the classes of the scheme.

THEOREM 10. *With the notation as above, suppose that $\mathcal{Q}_r$ is an association scheme on Ω_r admitting the group G_r, and that F_r is an inherent partition in $\mathcal{Q}_r$, for $r = 1, 2$. Then the crested product of the two groups preserves the crested product of the two association schemes (relative to the partitions F_1 and F_2 in each case).*

Proof. Since $\mathcal{Q}_1 \times \mathcal{Q}_2$ refines the crested product and is preserved by $G_1 \times G_2$, we only have to show that the group N^{F_1} preserves the crested product. To show this we consider separately the two types of relation which might hold between two pairs (α_1, α_2) and (β_1, β_2), and apply an element ϕ in N^{F_1} to both pairs.

If the matrix of the relation is $A_i \otimes B_j$ where $i \in \mathcal{L}$, then $F_1(\alpha_1) = F_1(\beta_1)$, and so the same element of N is applied to α_2 and β_2, so the B_j-relation they satisfy is preserved.

If the matrix is of the form $A_i \otimes D_m$, then the permutations $\phi(\alpha_1)$ and $\phi(\beta_1)$ do not change the F_2-parts of α_2 and β_2 respectively; but the D_m-relation depends only on these F_2-parts. □

We could have achieved Theorem 10 by being parsimonious about which permutations are included in the crested product of G_1 and G_2. The next result suggests that our definition is precisely the right one.

THEOREM 11. *Suppose that, for $r = 1, 2$, the classes of $\mathcal{Q}_r$ are the orbits of G_r on $\Omega_r \times \Omega_r$. Let F_1 be an inherent partition in $\mathcal{Q}_1$ and N a normal subgroup of G_2. Let F_2 be the orbit partition of N on Ω_2. Then F_2 is inherent in $\mathcal{Q}_2$, and the classes of the crested product $\mathcal{Q}$ of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with respect to F_1 and F_2 are the orbits of the crested product G of G_1 and G_2 on $(\Omega_1 \times \Omega_2) \times (\Omega_1 \times \Omega_2)$.*

Proof. Since G_2 centralizes R_{F_2}, the corresponding relation is a union of classes in $\mathcal{Q}_2$, so F_2 is inherent in $\mathcal{Q}_2$. Thus the crested product of the association schemes is defined. Even though N need not be the whole partwise stabilizer of F_2, the crested product of the groups is defined. Now Theorem 10 shows that each class of $\mathcal{Q}$ is a union of orbits of G, so we have to show that G is transitive on each class.

Suppose that $((\alpha_1, \alpha_2), (\beta_1, \beta_2))$ and $((\gamma_1, \gamma_2), (\delta_1, \delta_2))$ are in the same class of $\mathcal{Q}$. Then (α_1, β_1) and (γ_1, δ_1) are in the same class of $\mathcal{Q}_1$, so there is an element of the top group G_1 taking α_1 to γ_1 and β_1 to δ_1. Hence we may assume that $\alpha_1 = \gamma_1$ and $\beta_1 = \delta_1$.

If the class of $\mathcal{Q}$ has relation matrix $A_i \otimes B_j$ then (α_2, β_2) and (γ_2, δ_2) are in the same class of $\mathcal{Q}_2$. The crested product contains G_2 acting diagonally, and there is an element of G_2 taking (α_2, β_2) to (γ_2, δ_2).

Otherwise, the class of $\mathcal{Q}$ has relation matrix $A_i \otimes D_m$ for i not in $\mathcal{L}$, so α_1 and β_1 are not in the same part of F_1. There is some j in $\mathcal{K}_2$ for which D_m is a scalar multiple of $B_j R_{F_2}$. Hence there are points ε and ζ in Ω_2 such that (α_2, ε) and (γ_2, ζ) are in the B_j-relation while (ε, β_2) and (ζ, δ_2) are pairs in the same parts of F_2. There is an element ϕ_0 in G_2 taking (α_2, ε) to (γ_2, ζ), and elements h_1 and h_2 in N with $\varepsilon h_1 = \beta_2$ and $\zeta h_2 = \delta_2$. Since α_1 and β_1 are in different parts of F_1, there are elements ϕ_1 and ϕ_2 of N^{F_1} with $\phi_1(\alpha_1) = \phi_2(\alpha_1) = 1$, $\phi_1(\beta_1) = h_1^{-1}$ and $\phi_2(\beta_1) = h_2$. Now G contains $\phi_1\phi_0\phi_2$ and $((\alpha_1, \alpha_2), (\beta_1, \beta_2))\phi_1\phi_0\phi_2 = ((\alpha_1, \alpha_2), (\beta_1, \varepsilon))\phi_0\phi_2 = ((\alpha_1, \gamma_2), (\beta_1, \zeta))\phi_2 = ((\alpha_1, \gamma_2), (\beta_1, \delta_2))$. □

REMARK. In fact, we can prove something a little stronger. A *homogeneous coherent configuration* [**14**] is a generalization of association scheme in which the symmetry condition is weakened to the condition that the transpose of every adjacency matrix is also an adjacency matrix. The crested product of homogeneous coherent configurations can be defined in just the same way as for association schemes. If G_r is transitive on Ω_r then the orbits of G_r on $\Omega_r \times \Omega_r$ form a homogeneous coherent configuration. The proofs of Theorems 10 and 11 show that the orbit partition of the crested product of two transitive groups is the crested product of the orbit partitions of the two groups.

REMARK. If the groups G_1 and G_2 have regular abelian subgroups A_1 and A_2, then the condition (implicit in Theorem 10) that the inherent partition F_2 is the orbit partition of a normal subgroup of G_2 is automatically satisfied. Also, the crested product has a regular abelian subgroup $A_1 \times A_2$. In particular, the association scheme has a dual [**13**]. Theorem 9 shows that this is the crested product in reverse order of the duals of the two schemes (using the dual inherent partitions).

REMARK. There is another similar permutation group, obtained by taking instead the base group to be the product of N^{Ω_1} and $G_2^{F_1}$. Their intersection is N^{F_1}, and again the top group normalizes the base group. The new group is equal to the direct product if $F_2 = E_2$ and $F_1 = U_1$, and is equal to the wreath product if $F_2 = U_2$ or if $F_1 = E_1$. This suggests that there might be a different sort of crested product of association schemes, so that the analogue of the preceding theorem holds. In Section 8 we define a more general product which has both of these as special cases.

8. *A more general product of association schemes*

Let $\mathcal{F}$ be a set of inherent partitions of $\mathcal{Q}_1$ which contains U_1 and is closed under $\wedge$ and $\vee$ (except that we do not require that $\mathcal{F}$ contain the empty supremum E). For each F in $\mathcal{F}$, let $\mathcal{L}_F$ be the set of indices in $\mathcal{K}_1$ such that A_i is a component relation of R_F. Use Möbius inversion to obtain $\mathcal{J}_F$ as the set of i such that F is minimal subject to $i \in \mathcal{L}_F$. The sets $\mathcal{J}_F$, for F in $\mathcal{F}$, partition $\mathcal{K}_1$, because $\mathcal{F}$ contains U_1 and is closed under $\wedge$. Write F_0 for the unique minimal partition in $\mathcal{F}$.

Further, suppose that ψ is a map from $\mathcal{F}$ to the set of inherent partitions of $\mathcal{Q}_2$ with the properties that $\psi(F_0) = E_2$ and ψ preserves order and suprema. Note that ψ need not be one-to-one nor preserve infima. For F in $\mathcal{F}$, write $\mathcal{D}_F$ for the set of adjacency matrices of the ideal partition $\vartheta(\psi(F))$.

The adjacency matrices of the *extended crested product* $\mathcal{Q}$ of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with respect to $\mathcal{F}$ and ψ are

$$A_i \otimes D \qquad \text{for } i \text{ in } \mathcal{J}_F \text{ and } D \text{ in } \mathcal{D}_F$$

for all F in $\mathcal{F}$. Clearly they are all symmetric. Since $\sum_{D\in\mathcal{D}_F} D = J_{\Omega_2}$, their sum is $\sum_{i\in\mathcal{K}_1} A_i \otimes J_{\Omega_2}$, which is $J_{\Omega_1\times\Omega_2}$. We have $I_{\Omega_1} = A_0 \in \mathcal{J}_{F_0}$ and $I_{\Omega_2} = B_0 \in \mathcal{D}_{F_0} = \{B_j : j \in \mathcal{K}_2\}$ because $\psi(F_0) = E_2$, so $\mathcal{Q}$ contains $I_{\Omega_1\times\Omega_2}$. If $G \preccurlyeq F$ then $\psi(G) \preccurlyeq \psi(F)$ so $\mathcal{A}_2|^{\psi(F)} \leqslant \mathcal{A}_2|^{\psi(G)}$ and therefore the span $\mathcal{A}$ of the adjacency matrices is $\sum_{F\in\mathcal{F}} \left[\mathcal{A}_1|_F \otimes \mathcal{A}_2|^{\psi(F)}\right]$. Consider partitions F and G in $\mathcal{F}$. The subalgebras of the subschemes are ordered in the same way as the inherent partitions, so $\mathcal{A}_1|_F\mathcal{A}_1|_G \subseteq \mathcal{A}_1|_{F\vee G}$. Also $\mathcal{A}_2|^{\psi(F)}\mathcal{A}_2|^{\psi(G)}$ is the ideal of $\mathcal{A}_2$ generated by $R_{\psi(F)}R_{\psi(G)}$, which is proportional to $R_{\psi(F)\vee\psi(G)}$, which is $R_{\psi(F\vee G)}$, so $\mathcal{A}_2|^{\psi(F)}\mathcal{A}_2|^{\psi(G)} = \mathcal{A}_2|^{\psi(F\vee G)}$. Hence $\mathcal{A}$ is closed under multiplication and so $\mathcal{Q}$ is an association scheme.

Here are some special cases of the extended crested product.

(1) $\mathcal{F} = \{U_1\}$, $\psi(U_1) = E_2$ gives the direct product.
(2) $\mathcal{F} = \{E_1, U_1\}$, $\psi(X_1) = X_2$ for $X \in \{E, U\}$ gives the wreath product.
(3) $\mathcal{F} = \{F_1, U_1\}$, $\psi(F_1) = E_2$, $\psi(U_1) = F_2$ gives the simple crested product.
(4) $\mathcal{F} = \{E_1, F_1, U_1\}$, $\psi(X_1) = X_2$ for $X \in \{E, F, U\}$ gives the product hinted at in the final remark in Section 7.

It is curious that the 'non-standard' crested product has a neater definition in these terms than the 'standard' one does.

This more general version of the crested product still has $E_1 \times U_2$ as an inherent partition whose subschemes are isomorphic to $\mathcal{Q}_2$ and whose quotient scheme is isomorphic to $\mathcal{Q}_1$. Nonetheless, there are, in general, many extensions of $\mathcal{Q}_2$ by $\mathcal{Q}_1$ that do not arise as crested products. Any Latin square of order n is an extension of $\underline{\underline{n}}$ by $\underline{\underline{n}}$ but is not a crested product. See [**18**] for many more examples.

If $\mathcal{Q}_r$ is the association scheme of an orthogonal block structure $\mathcal{G}_r$ for $r = 1, 2$ then the extended crested product of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with respect to $\mathcal{F}$ and ψ is the association scheme of the orthogonal block structure

$$\bigcup_{F\in\mathcal{F}} \{H_1 \times H_2 : H_1 \in \mathcal{G}_1,\ H_2 \in \mathcal{G}_2,\ H_1 \preccurlyeq F,\ H_2 \succcurlyeq \psi(F)\}\,.$$

It is not immediately obvious that this is closed under $\wedge$. However, suppose that $H_1 \preccurlyeq F \in \mathcal{F}$, $K_1 \preccurlyeq G \in \mathcal{F}$, $\psi(F) \preccurlyeq H_2$ and $\psi(G) \preccurlyeq K_2$. Then $H_1 \wedge K_1 \preccurlyeq F \wedge G \in \mathcal{F}$, and $\psi(F \wedge G) \preccurlyeq \psi(F) \wedge \psi(G) \preccurlyeq H_2 \wedge K_2$, because ψ preserves order: hence $(H_1 \wedge K_1) \times (H_2 \wedge K_2)$ is in the orthogonal block structure.

EXAMPLE 5. Take four association schemes $\mathcal{Q}_{rs}$ for r, s in $\{1, 2\}$. For $r = 1$, 2, the direct product $\mathcal{Q}_{r1} \times \mathcal{Q}_{r2}$ has inherent partitions $E_{r1} \times E_{r2}$, $E_{r1} \times U_{r2}$, $U_{r1} \times E_{r2}$ and $U_{r1} \times U_{r2}$. Take $\psi(X_{1s} \times Y_{1s}) = X_{2s} \times Y_{2s}$ for s in $\{1, 2\}$ and X, Y in $\{E, U\}$. Then the extended crested product of $\mathcal{Q}_{11} \times \mathcal{Q}_{12}$ and $\mathcal{Q}_{21} \times \mathcal{Q}_{22}$ with respect to ψ is $(\mathcal{Q}_{11}/\mathcal{Q}_{21}) \times (\mathcal{Q}_{12}/\mathcal{Q}_{22})$, so this construction does seem to precisely capture the idea of putting F above $\psi(F)$ but no more.

EXAMPLE 6. For $r = 1$, 2, let $\mathcal{Q}_r$ be the association scheme of the Latin square orthogonal block structure whose non-trivial partitions are R_r, C_r and L_r. Put $\mathcal{F} = \{E_1, L_1, U_1\}$ and $\psi(X_1) = X_2$ for X in $\{E, L, U\}$. The extended crested product is shown in Figure 3.

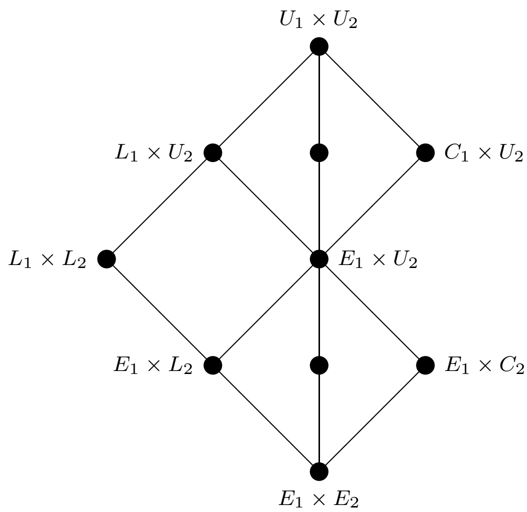


FIGURE 3. *Extended crested product of two Latin squares*

As before, the height of the lattice of the extended crested product of two orthogonal block structures is the sum of the original two heights. Examination of the small number of possible cases shows that the orthogonal block structure in Figure 3 cannot be attained from smaller structures by the simple crested product. Hence the extension gives something genuinely new.

9. *A more general product of permutation groups*

We can make a similar definition of the extended crested product of two permutation groups (G_1, Ω_1) and (G_2, Ω_2).

Let $\mathcal{F}$ be a set of G_1-invariant partitions of Ω_1.

Let $\mathcal{N}$ be a set of normal subgroups of G_2, closed under intersection (in particular, $\mathcal{N}$ contains the empty intersection G_2), and let ρ be an order-preserving map from $\mathcal{F}$ onto $\mathcal{N}$. For N in $\mathcal{N}$, let $F(N)$ be the orbit partition of N. If N_1 and N_2 are in $\mathcal{N}$ then $F(N_1)$ is orthogonal to $F(N_2)$ and $F(N_1) \vee F(N_2)$ is the orbit partition of N_1N_2. Even if N_i is the whole partwise stabilizer of $F(N_i)$ for $i = 1$, 2, the partwise stabilizer of $F(N_1) \vee F(N_2)$ may be larger than N_1N_2, so in general we do not insist on this condition.

Now the base group B of the extended crested product is defined to be the group generated by the subgroups $\rho(F)^F$ for $F \in \mathcal{F}$. (As usual, $\rho(F)^F$ is embedded in $G_2^{\Omega_1}$ as the set of functions constant on parts of F and taking values in $\rho(F)$.)

The extended crested product is generated by the base group and the group G_1 acting coordinatewise, as before. It is clear that each subgroup $\rho(F)^F$ is normalized by G_1, so we do have a semi-direct product; but the structure of the base group is less clear.

EXAMPLE 7. Let Ω_1 be $\underline{\underline{2}} \times \underline{\underline{2}}$ and let G_2 be the dihedral group

$$\langle a, b \colon a^4 = b^2 = 1,\ bab = a^{-1} \rangle$$

acting on the corners of a square. Take F_1 and F_2 to be the partitions of Ω_1 into rows and columns respectively, with $\rho(F_1) = \langle a^2, b \rangle$ and $\rho(F_2) = \langle a^2, ab \rangle$.

Figure 4 shows elements ϕ_i in $\rho(F_i)^{F_i}$ for $i = 1$, 2, and their product $\phi_1\phi_2$. It is readily checked that there are no elements ϕ_i' in $\rho(F_i)^{F_i}$ for $i = 1$, 2 such that $\phi_2'\phi_1' = \phi_1\phi_2$. Hence these two subgroups do not commute, so their product is not a group.

$$\phi_1 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline b & b \\ \hline \end{array} \qquad \phi_2 = \begin{array}{|c|c|}\hline 1 & ab \\ \hline 1 & ab \\ \hline \end{array} \qquad \phi_1\phi_2 = \begin{array}{|c|c|}\hline 1 & ab \\ \hline b & a^3 \\ \hline \end{array}$$

FIGURE 4. *Non-commuting subgroups of the base group in Example 7*

Nonetheless, we can indeed prove that the base group B is the product, in a suitable order, of all the subgroups that generate it. We use the fact that every partial order can be embedded in a total order.

THEOREM 12. *Suppose that $|\mathcal{F}| = n$. Relabel the partitions in $\mathcal{F}$ by $1, \ldots, n$ such that $i \leqslant j$ whenever $F_i \preccurlyeq F_j$. Put $N_i = \rho(F_i)$ and $B_i = N_i^{F_i}$ for $i = 1, \ldots, n$. Then $B_1B_2 \ldots B_n$ is a group.*

Proof. Assume for induction that

(a) $B_1B_2 \ldots B_i$ is a group C_i;

(b) C_i commutes with B_j whenever $i < j \leqslant n$.

These statements are both true when $i = 0$, for then C_i is the trivial group.

Assume that both are true for all non-negative integers less than i. Statement (b) for $i-1$ shows that $C_{i-1}B_i$ is a group, so that (a) is true for i.

Consider the product C_iB_j, where $i < j \leqslant n$. If $F_i \preccurlyeq F_j$ then B_j normalizes B_i so $B_iB_j = B_jB_i$. Otherwise, since $\mathcal{N}$ is meet-closed and ρ is onto, there is some k with $1 \leqslant k \leqslant i-1$ and $F_k \preccurlyeq F_i \wedge F_j$ and $\rho(F_k) = N_i \cap N_j$. If $\phi_i \in B_i$ and $\phi_j \in B_j$ then ϕ_i and ϕ_j take constant values (in N_i and N_j respectively) throughout each part of F_k, and these values commute modulo $N_i \cap N_j$, which is N_k. Hence $B_kB_iB_j = B_kB_jB_i$. Use of (b) for $i-2$, $i-3$, …, k shows that $C_{i-1} = B_1B_2 \ldots B_{i-1} = B_{i-1}B_{i-2} \ldots B_{k+1}B_1B_2 \ldots B_k$. Therefore

$$\begin{aligned} C_iB_j = C_{i-1}B_iB_j &= B_{i-1}B_{i-2} \ldots B_{k+1}B_1B_2 \ldots B_kB_iB_j \\ &= B_{i-1}B_{i-2} \ldots B_{k+1}B_1B_2 \ldots B_kB_jB_i \\ &= C_{i-1}B_jB_i. \end{aligned}$$

In both cases, $C_iB_j = C_{i-1}B_jB_i$, which is equal to $B_jC_{i-1}B_i$ by (b) for $i-1$, which is B_jC_i. Thus statement (b) is true for i. □

Now it is easily checked that both crested products of groups mentioned in Section 7 are indeed produced by this construction: for the simple crested product $\mathcal{F} = \{F_1, U_1\}$, $\rho(F_1) = N$ and $\rho(U_1) = G_2$; while the 'alternative' crested product is obtained when $\mathcal{F} = \{E_1, F_1\}$, $\rho(E_1) = N$ and $\rho(F_1) = G_2$. Also, the extended crested product contains the direct product and is contained in the wreath product.

At first sight it seems strange that, for the group construction, $\mathcal{F}$ does not need to be closed under $\vee$ and $\wedge$. However, suppose that F_1 and F_2 are incomparable partitions in $\mathcal{F}$. Then $N_i^{F_i}$ contains $N_i^{F_1 \vee F_2}$ for $i = 1$, 2 and so the base group contains $(N_1N_2)^{F_1 \vee F_2}$. The orbits of $N_1 \cap N_2$ may be smaller than the parts of $F(N_1) \wedge F(N_2)$ but there is some k with $F_k \preccurlyeq F_1 \wedge F_2$ and $N_k = N_1 \cap N_2$, so the base group contains $(N_1 \cap N_2)^{F_k}$, which contains $(N_1 \cap N_2)^{F_1 \wedge F_2}$. Thus including $F_1 \vee F_2$ and $F_1 \wedge F_2$ in $\mathcal{F}$ and putting $\rho(F_1 \vee F_2) = N_1N_2$ and $\rho(F_1 \wedge F_2) = N_1 \cap N_2$ still satisfies all the conditions but adds nothing to the base group.

10. *Linking the groups to the schemes*

It is rather curious that the conditions on $\mathcal{F}$ and ψ needed to ensure that the extended crested product of association schemes is indeed an association scheme are slightly different from those on $\mathcal{F}$ and ρ which we require to ensure that the base group of the extended crested product of permutation groups is the product of its generating subgroups.

In order to link the extended crested product of permutation groups to the extended crested product of association schemes, we need to introduce an interesting operator on semi-lattices. If $\mathcal{F}$ is a meet-closed poset, define $F^{\perp}$, for F in $\mathcal{F}$, to be the infimum of all those K in $\mathcal{F}$ for which $K \not\preccurlyeq F$. By convention, the empty infimum is the relevant maximal element, so that if $\mathcal{F}$ consists of partitions and F is uniquely maximal in $\mathcal{F}$ then $F^{\perp} = U$, while if $\mathcal{F}$ consists of normal subgroups of a group G and F is uniquely maximal in $\mathcal{F}$ then $F^{\perp} = G$. The map $F \mapsto F^{\perp}$ preserves order and infima and has the property that, for any F_1, F_2 in $\mathcal{F}$, either $F_1 \preccurlyeq F_2$ or $F_2^{\perp} \preccurlyeq F_1$. However, this map need not be one-to-one.

Dually, if $\mathcal{F}$ is join-closed then define $F^{\top}$, for F in $\mathcal{F}$, to be the supremum of all those K in $\mathcal{F}$ for which $K \not\succcurlyeq F$. Then the map $F \mapsto F^\top$ preserves order and suprema, and either $F_1 \preccurlyeq F_2$ or $F_2 \preccurlyeq F_1^\top$.

LEMMA 13. *Let $\mathcal{F}$ be a lattice.*
(a) *If $F \in \mathcal{F}$ and $F \not\preccurlyeq F^\top$ then $F^{\top\perp} = F$.*
(b) *If $\mathcal{F}$ is distributive and $F \in \mathcal{F}$ and F is join-irreducible then $F \not\preccurlyeq F^\top$.*

Proof.
(a) For K in $\mathcal{F}$, if $K \not\preccurlyeq F^\top$ then $F \preccurlyeq K$. Hence $F \preccurlyeq F^{\top\perp}$. If $F \not\preccurlyeq F^\top$ then $F^{\top\perp} \preccurlyeq F$. Therefore $F^{\top\perp} = F$.
(b) By definition, $F^\top = \bigvee\{K : K \not\succcurlyeq F\}$, so distributivity implies that $F \wedge F^\top = \bigvee\{F \wedge K : K \not\succcurlyeq F\}$. If $K \not\succcurlyeq F$ then $F \wedge K \neq F$, so $\bigvee\{F \wedge K : K \not\succcurlyeq F\}$ cannot be equal to F, because F is join-irreducible. Hence $F \wedge F^\top \neq F$, and so $F \not\preccurlyeq F^\top$.

□

In order that the extended crested product of groups preserve the extended crested product of the association schemes, we need the sets of conditions for the two products to be defined, and a link between ψ and ρ. We obtain the link by applying the operator $^\perp$ to the set of normal subgroups of G_2 which are the partwise stabilizers of the partitions $\psi(F)$.

THEOREM 14. *Suppose that $\mathcal{F}_r$ is a set of inherent partitions in the association scheme $\mathcal{Q}_r$ on Ω_r, and that the permutation group G_r preserves the classes of $\mathcal{Q}_r$, for $r = 1, 2$. Suppose that $\mathcal{F}_1$ contains U_1 and is closed under $\wedge$ and $\vee$, and that ψ is a map from $\mathcal{F}_1$ to $\mathcal{F}_2$ which preserves order and suprema and which carries the unique minimal element F_0 of $\mathcal{F}_1$ to E_2. For F in $\mathcal{F}_1$, suppose that there exists a normal subgroup $N_{\psi(F)}$ of G_2 whose orbit partition is $\psi(F)$, and let $\rho(F)$ be the intersection of the $N_{\psi(H)}$ for H in $\mathcal{F}_1$ with $H \not\preccurlyeq F$. Then the crested product of G_1 and G_2 with respect to ρ preserves the classes of the crested product of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with respect to ψ.*

Proof. Put $\mathcal{N} = \rho(\mathcal{F}_1)$. Then $G_2 = \rho(U_1) \in \mathcal{N}$ and $\rho(F) \cap \rho(H) = \rho(F \wedge H)$ for F, H in $\mathcal{F}_1$, so ρ satisfies the conditions to define the crested product of permutation groups.

The argument for preserving the association scheme is similar to that for the simple crested product. Suppose that the pairs (α_1, α_2) and (β_1, β_2) satisfy the relation with matrix $A_i \otimes D$, where $i \in \mathcal{J}_F$ and $D \in \mathcal{D}_F$, and take ϕ in $(\rho(H))^H$; we must show that $(\alpha_1, \alpha_2\phi(\alpha_1))$ and $(\beta_1, \beta_2\phi(\beta_1))$ satisfy the same relation.

If $F \preccurlyeq H$, then $\phi(\alpha_1) = \phi(\beta_1)$, which is in G_2 and preserves the D-relation. On the other hand, if $F \not\preccurlyeq H$, then $\rho(H)$ stabilizes $\psi(F)$ partwise; however, the D-relation depends only on the $\psi(F)$-classes. □

We now consider some examples.

In the case where $\mathcal{F}_1$ is a chain, so that $\rho(F)$ is the partwise stabilizer of the partition immediately above $\psi(F)$ for $F \neq U$, we see that if $F_1 \preccurlyeq F_2$ then $(\rho(F_2))^{F_2}$ normalizes $(\rho(F_1))^{F_1}$, so the base group is their product. We have seen that the simple crested product and the modified version in Section 7 are examples.

EXAMPLE 8. In these examples, we assume that the set of partitions in each of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy the appropriate order and meet relations in one of the diagrams in Figure 5. Moreover, we have partitions F_i for each label i in the diagram for $\mathcal{Q}_1$, partitions K_i for each label i in the diagram for $\mathcal{Q}_2$, and N_i is a normal subgroup of G_2 whose orbit partition is K_i. We also assume that $K_1 = E_1$ in each case, so that $N_1 = 1$, and that the coarsest partition shown is U.

First, take $\mathcal{Q}_1$ and $\mathcal{Q}_2$ both to have the inherent partitions in Figure 5(a), with the obvious isomorphism ψ. Then $\rho(F_i) = N_i^\perp = N_1$ for all $i \neq 5$, while $N_5^\perp = G_2$. Therefore the extended crested product of the groups is just the direct product, even though the extended crested product of the association schemes is not.

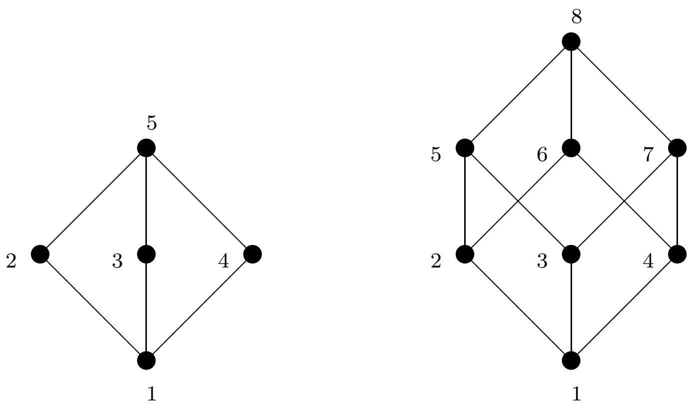


FIGURE 5. *Examples of extended crested products of groups*

Secondly, we can take $\mathcal{Q}_1$ and $\mathcal{Q}_2$ to be as in Figures 5(b) and (a) respectively, with $\psi(F_i) = K_i$ for $i = 1$, 2, 3, 4 and $\psi(F_i) = U$ otherwise. Then the base group of the extended crested product of the groups is just the product of $G_2^{F_5}$, $G_2^{F_6}$ and $G_2^{F_7}$: the partitions F_2, F_3, F_4, K_2, K_3, and K_4 contribute nothing to the group even though they play a role in the association scheme.

If we reverse the roles of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ then we can take $\psi(F_i) = K_{i+3}$ for $i = 2$, 3, 4 and find that the base group of the extended crested product of the groups is generated by $(N_6 \cap N_7)^{F_2}$, $(N_5 \cap N_7)^{F_3}$, $(N_5 \cap N_6)^{F_4}$ and G_2^U; the subgroups N_5, N_6 and N_7 do not appear in the expression for the base group.

However, if we take both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ to be as in Figure 5(b), with the obvious isomorphism ψ, then, as we shall show in the next theorem, the group and the association scheme match perfectly, and we obtain a generalization of Theorem 11.

THEOREM 15. *Suppose that, in addition to the hypotheses of Theorem 14, the following hold:*

(i) *$\mathcal{F}$ is distributive,*

(ii) *ψ is a lattice isomorphism,*

(iii) *$N_{\psi(F_1)} \cap N_{\psi(F_2)} = N_{\psi(F_1)\wedge\psi(F_2)}$ and $N_{\psi(F_1)}N_{\psi(F_2)} = N_{\psi(F_1)\vee\psi(F_2)}$, for all F_1 and F_2 in $\mathcal{F}$, and*

(iv) *the orbits of G_r on $\Omega_1 \times \Omega_2$ are the classes of $\mathcal{Q}_r$ for $r = 1$, 2.*

Then the orbits of G on $\Omega \times \Omega$ are the classes of $\mathcal{Q}$.

Proof. Since we have already shown that G preserves the classes of $\mathcal{Q}$, all we need to do is show that each class of $\mathcal{Q}$ is a single orbit of G on $\Omega \times \Omega$.

Consider the class with adjacency matrix $A_i \otimes D$, where $i \in \mathcal{J}_F$ and $D \in \mathcal{D}_F$ for some F in $\mathcal{F}$. Suppose that $((\alpha_1, \beta_1), (\gamma_1, \delta_1))$ and $((\alpha_2, \beta_2), (\gamma_2, \delta_2))$ are both in this class. There is an element of the top group which takes α_1 to α_2 and γ_1 to γ_2, so we may suppose that $\alpha_1 = \alpha_2 = \alpha$ and $\gamma_1 = \gamma_2 = \gamma$.

If F is the minimal element of $\mathcal{F}$ then $\psi(F) = E_2$ so D is just an adjacency matrix for $\mathcal{Q}_2$ and so there is some g in G_2 with $(\beta_1, \delta_1)g = (\beta_2, \delta_2)$.

If F is not the minimal element of $\mathcal{F}$ then there is a positive integer n and join-irreducibles $H_1, \ldots, H_n$ in $\mathcal{F}$ such that $F = H_1 \vee \ldots \vee H_n$. For $m = 1, \ldots, n$, Lemma 13(b) shows that $H_m \not\preccurlyeq H_m^\top$. Write N_m for $N_{\psi(H_m)}$. Because ψ and the map $\psi(K) \mapsto N_{\psi(K)}$ are both lattice isomorphisms, Lemma 13(a) shows that $\rho(H_m^\top) = N_m$, so that the base group contains $N_m^{H_m^\top}$.

Since (β_1, δ_1) and (β_2, δ_2) are both in the D-relation, there is some j in $\mathcal{K}_2$ and points ε_1, ε_2 in Ω_2 such that (β_1, ε_1) and (β_2, ε_2) are in the B_j-class of $\mathcal{Q}_2$ while ε_r is in the same part of $\psi(F)$ as δ_r for $r = 1$, 2. Now, $\psi(F) = \psi(H_1) \vee \ldots \vee \psi(H_n)$, and the matrices $R_{\psi(H_m)}$ commute pairwise, so there are elements $\zeta_{r0}, \zeta_{r1}, \ldots, \zeta_{rn}$ in Ω_2, for $r = 1$, 2, such that $\varepsilon_r = \zeta_{r0}$, $\delta_r = \zeta_{rn}$, and $\zeta_{r,m-1}$ and $\zeta_{r,m}$ are in the same part of $\psi(H_m)$ for $m = 1, \ldots, n$.

The extended crested product G of the groups contains G_2 acting diagonally, so it contains an element ϕ_0 taking (β_1, ε_1) to (β_2, ε_2).

If α and γ are in the same part of $H_m^\top$, then the definition of $\mathcal{J}_F$ shows that $H_m^\top \succcurlyeq F \succcurlyeq H_m$, which contradicts Lemma 13(b) for H_m. Hence α and γ are in different parts of $H_m^\top$ for $m = 1, \ldots, n$. For $r = 1$, 2 and $m = 1, \ldots, n$ there is an element g_{rm} in N_m such that $\zeta_{r,m-1} g_{rm} = \zeta_{r,m}$. Hence there are elements ϕ_{1m} and ϕ_{2m} in $N_m^{H_m^\top}$ such that $\phi_{1m}(\alpha)$ and $\phi_{2m}(\alpha)$ are the identity, $\phi_{1m}(\gamma) = g_{1m}^{-1}$ and $\phi_{2m}(\gamma) = g_{2m}$ for $m = 1, \ldots, n$. Now $\beta_1 \phi_{1n}(\alpha) \ldots \phi_{11}(\alpha) \phi_0 \phi_{21}(\alpha) \ldots \phi_{2n}(\alpha) = \beta_2$ and $\delta_1 \phi_{1n}(\gamma) \ldots \phi_{11}(\gamma) \phi_0 \phi_{21}(\gamma) \ldots \phi_{2n}(\gamma) = \delta_2$. □

11. *Remarks and problems*

We have not attempted to calculate the character table of a generalized crested product, or to give conditions for the generalized crested product to have a formal dual.

Another generalization of direct and wreath product in the literature is that of the *generalized wreath product* [**4**, **5**]. We do not define this here; but note that it is a product of a family of association schemes (or permutation groups) indexed by a partially ordered set, that it reduces to direct or wreath product in case the partially ordered set is a two-element antichain or chain respectively, and that the generalized wreath product of trivial association schemes is the scheme derived from a poset block structure (using the same poset).

One could now try to combine the two constructions. There are two possible ways in which this combination could take place. First, we could take the (extended) crested product of generalized wreath products. As noted above, a generalized wreath product of trivial schemes is a poset block structure, and we have already seen that a crested product of poset block structures is a poset block structure (so at least in this case we obtain nothing new).

The other possible combination is far more speculative. Thinking of a generalized wreath product as built from the ingredients "direct product" and "wreath product", applied to schemes whose indices are incomparable or comparable respectively, we could try to replace these ingredients by the more general notion of crested product. It is not even clear what kind of mathematical structure should replace the poset on the index set of the association schemes in order to describe such a construction!

An important question we have not considered is that of counting orbits of permutation groups on n-tuples of points (or more generally, of calculating the cycle index of permutation groups). We refer to [**10**] for the cycle index of the direct and wreath products (the first of these is, of course, well known). Also, in the paper [**16**], the number of orbits on n-tuples of a generalized wreath product of symmetric groups is calculated. There are many open problems here.

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