Intersecting Families of Permutations

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Abstract

Let S_n be the symmetric group on the set $X = \{1,2,.....,n\}$. A subset S of S_n is *intersecting* if for any two permutations g and h in S, $g(x) = h(x)$ for some $x \in X$ (that is g and h *agree* on x). M. Deza and P. Frankl [4] proved that if $S \subseteq S_n$ is intersecting then $|S| \leq (n-1)!$. This bound is met by taking S to be a coset of a stabilizer of a point. We show that these are the only largest intersecting sets of permutations.

1 Introduction

The following theorem is proved by M. Deza and P. Frankl in [4]:

Theorem 1 *Let S be an intersecting set of permutations of $\{1,\ldots,n\}$. Then $|S| \leq (n-1)!$.*

Our main result is the following:

Theorem 2 *Let $n \geq 2$ and $S \subseteq S_n$ be an intersecting set of permutations such that $|S| = (n-1)!$. Then S is a coset of a stabilizer of one point.*

Suppose that the set S satisfying the conditions in Theorem 2 does not contain the identity element ι. Then taking a permutation $g \in S$, $S' = g^{-1}S = \{g^{-1}h : h \in S\}$ now contains ι and again satisfies the conditions in Theorem 2. Hence, assuming $\iota \in S$, it is enough to show that S is a stabilizer of one point.

For each $g \in S_n$, we say that a point x is *fixed* by g if $g(x) = x$. The set $\mathrm{Fix}(g) = \{x \in X \ : \ g(x) = x\}$ is the *fixed point set* of g. Moreover if S is a subset of S_n, then $\mathrm{Fix}(S) = \{\mathrm{Fix}(g) \ : \ g \in S\}$ is a family of subsets of X.

Let $x \in X$, $g \in S_n$. We define the *fixing* of the point x via g to be the permutation $g_x \in S_n$ such that

(i) If $g(x) = x$, then $g_x = g$;

(ii) If $g(x) \neq x$, then

$$g_x(y) = \begin{cases} x & \text{if } y = x, \\ g(x) & \text{if } y = g^{-1}(x), \\ g(y) & \text{if } y \neq x \text{ , } y \neq g^{-1}(x). \end{cases}$$

Inductively we define $g_{x_1,\dots,x_q}$ to be the fixing of x_q via $g_{x_1,\dots,x_{q-1}}$. We also say that a set of permutations S is *closed under the fixing operation* if the following holds:

$$\text{for each } x \in X \text{ and } g \in S, g_x \in S.$$

Using GAP [6], it is not difficult to establish our theorem if $n \leq 5$. So we may assume that $n \geq 6$. We now give the outline of our proof: we first show that a set of permutations S which satisfies the conditions in Theorem 2 is closed under the fixing operation (Theorem 8). This implies that $\mathrm{Fix}(S)$ is an intersecting family of subsets (that is $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) \neq \emptyset$ for any g, $h \in S$): this is the statement of Theorem 10. With these assumptions, we finally show that S must be a stabilizer of one point in section 5.

2 Preliminary results

A graph is *vertex-transitive* if any vertex can be mapped into any other by an automorphism. A subgraph of a graph is called *clique* if any two of its vertices are adjacent. A *coclique* is a subgraph in which no two vertices are adjacent.

Theorem 3 *Let Γ be a vertex transitive graph on n vertices. Suppose that T is a subset of the vertex set, and that the largest clique contained in T has size $|T|/m$. Then any clique S in Γ satisfies $|S| \leq n/m$. Equality implies that $|S \cap T| = |T|/m$.*

Proof Count pairs (v,g) with $v \in S$, $g \in Aut(\Gamma)$ and $g(v) \in T$. For each $w \in T$ there are $|Aut(\Gamma)|/n$ choices of g with $g(v) = w$; so the number of pairs is $|S| \cdot |Aut(\Gamma)|/n \cdot |T|$. On the other hand, for any automorphism g, we have $|g(S) \cap T| \leq |T|/m$ (since $g(S) \cap T$ is a clique in T); so the number of pairs is at most $|T|/m \cdot |Aut(\Gamma)|$. Thus

$$|S| \cdot |Aut(\Gamma)|/n \cdot |T| \leq |T|/m \cdot |Aut(\Gamma)|,$$

so

$$|S| \leq n/m.$$

If equality holds then $|g(S) \cap T| = |T|/m$ for all $g \in Aut(\Gamma)$. Taking $g = \iota$ gives the result. $\square$

If T is a coclique, then the largest clique it contains has size 1, so the hypothesis holds with $m = |T|$. This gives the following:

Corollary 4 *Let C be a clique and A a coclique in a vertex-transitive graph on n vertices. Then $|C| \cdot |A| \leq n$. Equality implies that $|C \cap A| = 1$.*

Theorem 5 *Let S be an intersecting set of permutations of $\{1, 2, \ldots\ldots, n\}$. Then $|S| \leq (n-1)!$. If equality holds, then S contains a row of each Latin square of order n.*

Proof Form a graph on the vertex set S_n by joining g and h if $g(i) = h(i)$ for some point i. It is clear that left multiplication by elements of S_n is an automorphism; so the graph is vertex-transitive. Let L be the set of rows of a Latin square. Then S is a clique and L is a coclique with $|L| = n$. So, by Proposition 4, $|S| \leq n!/n = (n-1)!$, and equality implies $|S \cap L| = 1$. $\square$

We need another definition before stating the next proposition. Let g be a permutation in S_n. We define

$$D(g) = \{w \in S_n : \ w(i) \neq g(i) \ \ \forall i = 1, \ldots, n\}$$

Proposition 6 *Let $n \geq 2k$. Then, for any $g_1, g_2, \ldots\ldots, g_k \in S_n$, we have $D(g_1) \cap D(g_2) \cap \ldots\ldots \cap D(g_k) \neq \emptyset$.*

Proof A permutation $h \in S_n$ belongs to $D(g_1) \cap D(g_2) \cap \ldots \cap D(g_k)$ if and only if it is a system of distinct representatives for the sets $A_1, \ldots, A_n$, where

$$A_i = \{x \,:\, x \neq g_1(i) \text{ and } x \neq g_2(i) \text{ and .....and } x \neq g_k(i)\}.$$

Clearly $|A_i| \geq n-k$.

We must check the conditions of Philip Hall's Marriage Theorem. Let $A(J) = \bigcup_{j \in J} A_j$ for $J \subseteq \{1, ..., n\}$. We must show that $|A(J)| \geq |J|$ for all J. Clearly this holds if $|J| \leq n-k$, so we can suppose that $|J| \geq n-k+1$.

Take $x \in \{1, ..., n\}$. Then $x \notin A(J)$ if and only if, for all $j \in J$, there exists $i \in \{1, ..., k\}$ such that $x = g_i(j)$. But there are at most k pairs (i, j) with $x = g_i(j)$, since given i, the value of j is determined ($j = g_i^{-1}(x)$). Since $|J| \geq n-k+1 \geq k+1$, this cannot hold for all $j \in J$. Thus $A(J) = \{1, ..., n\}$ and $|A(J)| = n \geq |J|$. $\square$

Remark If the permutations $g_1, ..., g_k$ are pairwise non-intersecting then the condition $n \geq 2k$ can be weakened to $n \geq k+1$. Hence any $k \times n$ Latin rectangle (set of pairwise non-intersecting permutations) can be extended to a Latin square: this is the result of Marshall Hall (Theorem 7). Let $g_1, \ldots, g_k$ be the rows of a Latin square of order k, extended to fix the points $k+1, \ldots, n$. Any permutation in $D(g_1) \cap \ldots \cap D(g_k)$ must have symbols from the set $k+1, \ldots, n$ in positions $1, \ldots, k$; so if $n \leq 2k-1$, then no such permutation can exist.

Theorem 7 [M. Hall 1945] *Every $k \times n$ latin rectangle can be extended to some $n \times n$ latin square.*

3 Closure under fixing operation

Let $g \in S_n$ and $A \subseteq X$. If $g(A) = A$, then the permutation g *restricted* to A, denoted by $g|_A$, is a bijection from A to itself, and so it is an element in $\mathrm{Sym}(A)$. However, in general, $g|_A$, being a bijection between $|A|$-subsets of X, is a *partial permutation*.

Theorem 8 *Let $S \subseteq S_n$ be an intersecting set of permutations such that $\iota \in S$ and $|S| = (n-1)!$ where $n \geq 6$. Then S is closed under the fixing operation.*

Proof Assume that S is not closed under the fixing operation. Then there exists some $x \in X$ and $g \in S$ such that $g(x) \neq x$ and $g_x \notin S$. Now let $g = a_1 a_2 .....a_x ...a_y ...a_n$ where $a_x \neq x$, $a_y = x$. So

$$g_x = a_1.....a_{x-1}a_y a_{x+1}.....a_{y-1}a_x a_{y+1}.....a_n.$$

We consider the following cases:

(i): $a_x = y$

Let $X \setminus \{x,y\} = A$. Then $\bar{\iota} = \iota|_A$ and $\bar{g} = g|_A = g_x|_A$ are elements in $\mathrm{Sym}(A)$. By Proposition 6, there exists $\bar{h} \in D(\bar{\iota}) \cap D(\bar{g})$ since $n-2 \geq 4$. Now construct a permutation h on X as follows:

$$h(i) = \begin{cases} \bar{h}(i) & \text{if } i \in A, \\ y & \text{if } i = x, \\ x & \text{if } i = y. \end{cases}$$

$$\begin{array}{ccccccccccc}
\iota & : & \cdots & x & \cdots & u & \cdots & y & \cdots \\
g & : & \cdots & y & \cdots & a_u & \cdots & x & \cdots \\
\bar{\iota} & : & \cdots & \blacksquare & \cdots & u & \cdots & \blacksquare & \cdots \\
\bar{g} & : & \cdots & \blacksquare & \cdots & a_u & \cdots & \blacksquare & \cdots \\
\bar{h} & : & \cdots & \blacksquare & \cdots & b_u & \cdots & \blacksquare & \cdots \\
h & : & \cdots & y & \cdots & b_u & \cdots & x & \cdots \\
g_x & : & \cdots & x & \cdots & a_u & \cdots & y & \cdots
\end{array}$$

Then g_x and h form a $2 \times n$ latin rectangle. By Theorem 7, there exists a $n \times n$ latin square containing g_x and h. But observe that for any row r in this latin square other than g_x and h, we must have $r \in D(g_x) \cap D(h)$ and hence $r \in D(g)$, that is r and g agree on no points in X. So $r \notin S$ since $g \in S$ and S is intersecting. Moreover h and ι also agree on no points in X by construction and thus $h \notin S$ since $\iota \in S$ and S is intersecting. Further $g_x \notin S$ by assumption. Hence no rows in this latin square lie in S. But this contradicts Theorem 5.

(ii): $a_x = z \neq y$

Let $A = X \setminus \{x,z\}$. So $\bar{\iota} = \iota|_A$ is the identity in $\mathrm{Sym}(A)$. Now define another permutation $\bar{g}$ on A as follows:

$$\bar{g}(i) = \begin{cases} g(i) & \text{if } i \neq y, \\ g(z) & \text{if } i = y. \end{cases}$$

But $|A| = n-2 \geq 4$, and so by Proposition 6, there exist a permutation $\overline{h} \in D(\bar{\iota}) \cap D(\overline{g}) \subseteq Sym(A)$. We now construct a permutation h_* on X as follows:

$$h_*(i) = \begin{cases} \overline{h}(i) & \text{if } i \in A, \\ z & \text{if } i = x, \\ x & \text{if } i = z. \end{cases}$$

We further construct a permutation h on X as follows:

$$h(i) = \begin{cases} h_*(i) & \text{if } i \neq y, z, \\ h_*(z) = x & \text{if } i = y, \\ h_*(y) & \text{if } i = z. \end{cases}$$

$$\begin{array}{cccccccccccc}
\iota & : & \cdots & x & \cdots & u & \cdots & y & \cdots & z & \cdots \\
g & : & \cdots & z & \cdots & a_u & \cdots & x & \cdots & a_z & \cdots \\
\bar{\iota} & : & \cdots & \blacksquare & \cdots & u & \cdots & y & \cdots & \blacksquare & \cdots \\
\overline{g} & : & \cdots & \blacksquare & \cdots & a_u & \cdots & a_z & \cdots & \blacksquare & \cdots \\
\overline{h} & : & \cdots & \blacksquare & \cdots & b_u & \cdots & b_y & \cdots & \blacksquare & \cdots \\
h_* & : & \cdots & z & \cdots & b_u & \cdots & b_y & \cdots & x & \cdots \\
h & : & \cdots & z & \cdots & b_u & \cdots & x & \cdots & b_y & \cdots \\
g_x & : & \cdots & x & \cdots & a_u & \cdots & z & \cdots & a_z & \cdots
\end{array}$$

We claim that g_x and h form a $2 \times n$ latin rectangle. It is readily checked that g_x and h do not agree on all the points in X except perhaps on z. But $h(z) = h_*(y) = \overline{h}(y)$ and $\overline{h} \in D(\overline{g})$ and therefore $h(z) \neq \overline{g}(y) = g(z) = g_x(z)$. This proves the claim. By Theorem 7, there exists a $n \times n$ latin square containing g_x and h.

Now observe that a row r in this latin square, other than g_x and h, if regarded as permutation, does not agree with g at any point in X. Moreover $g_x \notin S$ by assumption. So we are left to check if $h \in S$. By our construction, if h and ι were to agree on some point i, then $i \neq x$, y, z. But this would imply that $\overline{h}$ and $\bar{\iota}$ must agree on some point. But this is a contradiction since $\overline{h} \in D(\bar{\iota})$. Hence $h \notin S$. But this shows that no rows in this latin square lie in S contradicting Theorem 5.

Hence the theorem is proved. $\square$

4 Fixed point sets intersect

Lemma 9 *Let g, $h \in S_n$. Suppose that for some x, $y \in X$ where $g(x) = h(x)$ and $g(y) \neq h(y)$. Then $g_x(y) \neq h(y)$.*

Proof If $g(y) = x$ then $g_x(y) = g(x) = h(x) \neq h(y)$. If $g(y) = x$ then $g_x(y) = g(y) \neq h(y)$. $\square$

Theorem 10 *Let $S \subseteq S_n$ be an intersecting set of permutations which is closed under the fixing operation. Then* $\mathrm{Fix}(S)$ *is an intersecting family.*

Proof We claim that if $g, h \in S$ are such that $g(x) = h(x)$ and $g(y) \neq h(y)$ then $g_x(y) \neq h(y)$ and $g_x \in S$. This follows immediately from Lemma 9 and the fact that S is closed under the fixing operation.

Assume that $\mathrm{Fix}(S)$ is not intersecting. Then there are $g \neq h \in S$ such that $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) = \emptyset$. Let $B = \{x \in X : g(x) = h(x)\}$. Since S is intersecting, $B = \{x_1, \ldots, x_k\}$ for some positive integer k.

Let $w = g_{x_1 \ldots x_k}$. By the first paragraph, $w(y) \neq h(y)$ for every $y \in X \setminus B$, and $w \in S$. If $w(x_i)$ were equal to $h(x_i)$ for some i, we would have $x_i = w(x_i) = h(x_i) = g(x_i)$, where the last inequality follows from $x_i \in B$. But then $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) \neq \emptyset$, a contradiction. Hence $w(x) \neq h(x)$ for every $x \in X$. However, this is a contradiction with $w, h \in S$. $\square$

5 Proof of Theorem 2

We need the following well known results in extremal set theory:

Proposition 11 [LYM Inequality]
Let $\mathcal{A}$ be an antichain of subsets of an n-set X. Then

$$\sum_{A \in \mathcal{A}} |A|!(n - |A|)! \leq n!.$$

Proposition 12 [Erdős-Ko-Rado] *If $\{A_1, A_2, \ldots.., A_m\}$ is an intersecting family of k-subsets of an n-set X such that $k \leq n/2$, then*

$$m \leq \binom{n-1}{k-1}.$$

Lemma 13 *If $\mathcal{A}$ is an antichain of subsets of an $n-$set X such that $|A| \geq k$ for all $A \in \mathcal{A}$, then*

$$\sum_{A \in \mathcal{A}} (n - |A|)! \leq n!/k!.$$

Proof

$$\sum_{A\in\mathcal{A}}(n-|A|)! \leq \sum_{A\in\mathcal{A}}\frac{|A|!}{k!}(n-|A|)! = n!/k!,$$

by applying the LYM Inequality. □

We now give some observations.

Let $Y \subseteq X$ and $G = \mathrm{Sym}(X) = S_n$. We define $G_{(Y)}$ to be the set of all permutations $g \in S_n$ such that $g(y) = y$ for all $y \in Y$. Clearly $G_{(\{x\})}$ is the stabilizer of the point x and $|G_{(Y)}| = (n-|Y|)!$. Now if g is a permutation in S with the fixed point set $\mathrm{Fix}(g) = F$, then $g \in G_{(F)}$. Hence we deduce that

$$|S| \leq \sum_{F\in\mathrm{Fix}(S)}|G_{(F)}| = \sum_{F\in\mathrm{Fix}(S)}(n-|F|)!.$$

But we can do better. Observe that if $A \subseteq B$ for some $A,B \in \mathrm{Fix}(S)$, then $G_{(B)} \subseteq G_{(A)}$.

Hence taking

$$\mathcal{F} = \{F \in \mathrm{Fix}(S) : F \text{ is a minimal element in the poset } (\mathrm{Fix}(S),\subseteq)\},$$

we now have

$$|S| \leq \sum_{F\in\mathcal{F}}(n-|F|)!.$$

Proof of Theorem 2

Assuming $\iota \in S$, we want to show that S is a stabilizer of a point. We first note that the theorem is true for $n \leq 5$. This can be proved by hand or by computer using GAP [6]. (We are looking for cliques in the graph used in Theorem 5, which can be found using the clique finder in the GAP package GRAPE.) Let $n \geq 6$. By Theorem 8 and 10, we can now assume that $\mathrm{Fix}(S)$ is intersecting. Let $\mathcal{F}$ be the subset of $\mathrm{Fix}(S)$ as defined above. Then $\mathcal{F}$ now is an intersecting antichain of subsets of X and it is not empty.

Obviously $\emptyset \notin \mathcal{F}$ since $\mathcal{F}$ is intersecting. Moreover note that if a permutation g fixes more than $n-2$ points, then it must be the identity, and so $|\mathrm{Fix}(g)| \neq n-1$ for all $g \in S$, in particular, $|F| \neq n-1$ for all $F \in \mathcal{F}$. Also $X \notin \mathcal{F}$ since $\mathcal{F}$ is an antichain. Hence we have $1 \leq |F| \leq n-2$ for all $F \in \mathcal{F}$.

Suppose that $\mathrm{Fix}(S)$ contains an element of size 1, say $\{x\}$. Then by the intersection property of $\mathrm{Fix}(S)$, all permutations in S fix the point x. Since $|S| =$

$(n-1)!$, S now must be the stabilizer of x. So we can assume that $|\operatorname{Fix}(g)| \geq 2$ for all $g \in S$ and hence $|F| \geq 2$ for all $F \in \mathcal{F}$.

Moreover we can assume that $\bigcap_{F\in\mathcal{F}} F = \emptyset$, for otherwise, by the definition of $\mathcal{F}$, this implies that $\bigcap_{F\in\operatorname{Fix}(S)} F \neq \emptyset$, and hence all permutations in S fix a common point and the result again follows.

Having made the above simplifications, our aim is to derive a contradiction by showing that $|S| < (n-1)!$. We achieve this by considering the following cases:

Case (I) $|F| \geq 3$ for all $F \in \mathcal{F}$, that is, $\mathcal{F}$ has no element of size 2.

In this case, we have

$$\begin{aligned}
|S| &\leq \sum_{F\in\mathcal{F}} (n-|F|)! \\
&= \sum_{\substack{F\in\mathcal{F}\\ 3\leq|F|\leq[n/2]}} (n-|F|)! + \sum_{\substack{F\in\mathcal{F}\\ |F|\geq[n/2]+1}} (n-|F|)! \\
&\leq \sum_{k=3}^{[n/2]} a_k(n-k)! + \frac{n!}{([n/2]+1)!},
\end{aligned}$$

by Lemma 13, where a_k is the number of elements in $\mathcal{F}$ having the size k. Then

$$|S| \leq \sum_{k=3}^{[n/2]} \binom{n-1}{k-1}(n-k)! + \frac{n!}{([n/2]+1)!}$$

by the Erdős-Ko-Rado Theorem. So

$$\begin{aligned}
|S| &\leq (n-1)! \sum_{k=3}^{[n/2]} \frac{1}{(k-1)!} + \frac{n!}{([n/2]+1)!} \\
&\leq (n-1)! \cdot \frac{4}{5} + \frac{n!}{([n/2]+1)!},
\end{aligned}$$

since $\sum_{k=3}^{[n/2]} \frac{1}{(k-1)!} < e-2 < \frac{4}{5}$ where e is the natural exponent.

Hence it is enough to show that $\frac{n!}{([n/2]+1)!} < \frac{(n-1)!}{5}$. But this is true for $n \geq 8$. For $n = 6, 7$, it is readily checked from (1) that $|S| < (n-1)!$.

We conclude that if $\mathcal{F}$ has no element of size 2, then $|S| < (n-1)!$ for all $n \geq 6$.

Case (II) $\mathcal{F}$ contains an element of size 2.

Let $\mathcal{F}_2 = \{F \in \mathcal{F} \ : \ |F| = 2\}$.

Subcase (i) $\bigcap_{F\in\mathcal{F}_2} F = \emptyset$.

Without loss of generality, we can assume that $\{\{1,2\},\{1,3\},\{2,3\}\} \subseteq \mathcal{F}_2$ by the intersection property. Let $F \in \mathcal{F}\backslash\{\{1,2\},\{1,3\},\{2,3\}\}$. Since $F \cap \{2,3\} \neq \emptyset$, we have either $2 \in F$ or $3 \in F$. So this implies that $1 \notin F$ for otherwise $\{1,2\} \subseteq F$ or $\{1,3\} \subseteq F$ contradicts the antichain property of $\mathcal{F}$. But now $F \cap \{1,2\} \neq \emptyset$ and $F \cap \{1,3\} \neq \emptyset$ implies that $\{2,3\} \subseteq F$ contradicting that $\mathcal{F}$ is an antichain. Hence $\mathcal{F} = \mathcal{F}_2$ and we deduce that $|S| \leq \sum_{F\in\mathcal{F}}(n-|F|)! = \sum_{F\in\mathcal{F}_2}(n-|F|)! = 3(n-2)! < (n-1)!$ for $n \geq 6$.

Subcase (ii) $\bigcap_{F\in\mathcal{F}_2} F \neq \emptyset$.

Without loss of generality, we can assume that $\mathcal{F}_2 = \{\{1,i\} | 2 \leq i \leq c\}$ for some $c \in \{2,3,\ldots,n\}$.

Now let

$$\begin{aligned}\mathcal{D} &= \{F \in \mathcal{F}\backslash\mathcal{F}_2 \ : \ 1 \notin F\},\\ \mathcal{E} &= \{F \in \mathcal{F}\backslash\mathcal{F}_2 \ : \ 1 \in F\}.\end{aligned}$$

If g is a permutation with its fixed point set $\mathrm{Fix}(g)$ containing F for some $F \in \mathcal{D}$, then $\mathrm{Fix}(g)$ contains $\{2,3,\ldots,c\}$ since $\mathcal{F}$ is intersecting. So $g \in G_{(\{2,3,\ldots,c\})}$.

If $c = n$, then $\mathcal{D}$ is empty for otherwise $\{2,3,\ldots..,n\} \subseteq F$ for any $F \in \mathcal{D}$ would imply that $|F| > n-2$ which is a contradiction. Hence $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{E}$ and so all F in $\mathcal{F}$ must contain 1, that is, $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$. But this again contradicts our earlier assumption. So $c \leq n-1$.

If $F \in \mathcal{E}$, then $\{1,x,y\} \subseteq F$ for some $x,y \notin \{2,3,\ldots..,c\}$ since $\mathcal{F}$ is an antichain. Hence there are at most $\binom{n-c}{2}$ choices for the unordered pair $\{x,y\}$. If g is a permutation with its fixed point set $F(g)$ containing F for some $F \in \mathcal{E}$, then $g \in G_{(\{1,x,y\})}$. We now deduce that

$$|S| \leq \sum_{F\in\mathcal{F}_2}(n-|F|)! + |G_{(\{2,3,\ldots..,c\})}| + \sum_{B\in\binom{X\backslash\{1,2,\ldots..,c\}}{2}} |G_{(\{1\}\cup B)}|$$

$$\leq \quad (c-1)(n-2)! + (n-c+1)! + \binom{n-c}{2}(n-3)!.$$

Assuming $3 \leq c \leq n-2$, we have $|S| \leq f(c)$ where $f(c) = c(n-2)! + \binom{n-c}{2}(n-3)!$. But $\frac{n-c}{2} < n-2$ implies that

$$\frac{(n-c)(n-c-1)}{2} < (n-2)(n-c-1),$$

since $n-c-1 > 0$. So

$$\begin{aligned} \binom{n-c}{2}(n-3)! &< (n-2)!(n-c-1) \\ f(c) &< (n-1)!, \end{aligned}$$

and hence $|S| < (n-1)!$ for $n \geq 6$.
If $c = n-1$, then

$$|S| \leq \sum_{F \in \mathcal{F}_2} (n-|F|)! + |G_{(\{2,3,\ldots\ldots,n-1\})}| = (n-2)(n-2)! + 2 < (n-1)!$$

for all $n \geq 6$.
We can now assume that $\mathcal{F}_2 = \{\{1,2\}\}$ for $n \geq 6$. Then $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_2$, where

$$\begin{aligned} \mathcal{B}_1 &= \{F \in \mathcal{F} \setminus \mathcal{F}_2 : 1 \in F\} \\ \mathcal{B}_2 &= \{F \in \mathcal{F} \setminus \mathcal{F}_2 : 2 \in F\} \end{aligned}$$

Observe that $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ since $\mathcal{F}$ is an antichain. Also for each $i = 1,2$, if $F \in \mathcal{B}_i$, then F contains the set $\{i,a,b\}$ where a, $b \in X \setminus \{1,2\}$.
Hence

$$\begin{aligned} |S| &\leq \sum_{F \in \mathcal{F}_2} (n-|F|)! + \sum_{\{a,b\} \in \binom{X \setminus \{1,2\}}{2}} |G_{(\{1,a,b\})}| + \sum_{\{a,b\} \in \binom{X \setminus \{1,2\}}{2}} |G_{(\{2,a,b\})}| \\ &\leq (n-2)! + 2 \cdot \binom{n-2}{2} \cdot (n-3)! \\ &\leq (n-2)(n-2)! < (n-1)!. \end{aligned}$$

We conclude that if $\mathcal{F}$ has an element of size 2, then $|S| < (n-1)!$ for $n \geq 6$.

Hence the result follows. $\square$

6 Open problems

Problem 1 What is the cardinality of the largest intersecting subset of S_n which is not contained in a coset of the stabilizer of a point, and what is the structure of such a set of maximum cardinality?

Consider the following set of permutations (for $n \geq 4$):

$$S^* = \{g \in S_n : g(1) = 1, g(i) = i \text{ for some } i > 2\} \cup \{t\},$$

where t is the transposition interchanging 1 and 2. Then S^* is clearly intersecting and is not contained in a coset of the stabilizer of a point. Moreover, S^* is a maximal intersecting set. It satisfies

$$|S^*| = (n-1)! - d(n-1) - d(n-2) + 1 \sim (1 - e^{-1})(n-1)!,$$

where $d(m)$ is the number of derangements in S_m.

We conjecture that, for $n \geq 6$, an intersecting subset not contained in a coset of a point stabiliser has size at most $(n-1)! - d(n-1) - d(n-2) + 1$, and that a set meeting this bound has the form gS^*h for some $g, h \in S_n$. Computation using GAP [6] shows that this is true for $n = 6$.

A weaker conjecture is that there exists $c > 0$ such that any intersecting set $S \subseteq S_n$ with $|S| \geq (1-c)(n-1)!$ is contained in a coset of the stabiliser of a point.

Problem 2 Given $t \geq 1$, is there a number $n_0(t)$ such that, if $n \geq n_0(t)$, then a t-intersecting subset of S_n has cardinality at most $(n-t)!$, and that a set meeting the bound is a coset of the stabilizer of t points? (A set S of permutations is said to be t-intersecting if $|\{x : g(x) = h(x)\}| \geq t$ for any $g, h \in S$.)

Deza and Frankl [4] showed that the bound $(n-t)!$ holds if there exists a sharply t-transitive set of permutations of $\{1, \ldots, n\}$. (This is an immediate consequence of Corollary 4.) This holds, for example, if $t = 2$ and n is a prime power. Even in this special case, however, our argument for identifying a set meeting the bound fails, because there is no analogue of Hall's theorem for sharply t-transitive sets with $t > 1$.

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