Transitive permutation groups without semiregular subgroups

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Abstract

A transitive finite permutation group is called *elusive* if it contains no non-trivial semiregular subgroup. The purpose of this paper is to collect known information about elusive groups. The main results are recursive constructions of elusive permutation groups, using various product operations and affine group constructions. We also include a brief historical introduction and a survey of known elusive groups. In a sequel, Giudici determines all the quasiprimitive elusive groups.

Part of the motivation for studying this class of groups is a conjecture due to Marušič, Jordan and Klin, asserting that there is no elusive 2-closed permutation group. We show that the constructions given here will not build counterexamples to this conjecture.

1 Introduction

A permutation group (G, Ω), assumed throughout this paper to be finite, is called *elusive* if G is transitive and contains no non-trivial semiregular subgroup (equivalently, no fixed-point-free element of prime order). The name is intended to suggest that such groups are not so easy to find. It also suggests that, given such a group G, we lack one of the standard tools for studying the G-invariant graphs, namely, taking quotients by semiregular subgroups (cf. [2, 22]).

While every transitive permutation group contains an element of prime power order without fixed points (see [9, Theorem 1], which was motivated by an application to Brauer groups of local fields), the result is no longer true if "prime power" is replaced with "prime" in the above statement, as we will see.

A permutation group (G, Ω) is called 2-*closed* if every permutation of Ω that preserves the G-orbits on Ω^2 belongs to G [8, 24]. Note that the full automorphism group of any graph or digraph is 2-closed; conversely, every 2-closed group is the automorphism group of an edge-coloured digraph. (Not every transitive 2-closed group is the automorphism group of a graph or digraph. For example, consider the Klein group V_4 acting regularly. The V_4-invariant digraphs are all undirected and have automorphism group either D_8 or S_4.)

A permutation group contains nonidentity semiregular elements, that is, with all orbits of the same length, if and only if it contains elements of prime order with no fixed points. The problem of the existence of such elements in a 2-closed transitive permutation group was originally proposed in graph-theoretic language. See, for example, Biggs [2] for the use of semiregular automorphisms to give concise descriptions of interesting graphs.

In 1981, Marušič asked if there exists a vertex-transitive graph without a nonidentity semiregular automorphism [18, Problem 2.4]. It was also proved there that, given a prime p, every vertex-transitive graph of order p^k or of order mp, where $m \le p$, has an automorphism of order p with no fixed vertices. The proof of both of these results — the first being really just an observation — works in a more general setting for transitive permutation groups as well. Marušič and Scapellato [21] proved that every cubic vertex-transitive graph and every vertex-transitive graph of order $2p^2$ has an automorphism of prime order without fixed vertices; their proof applies to any 2-closed group of degree $2p^2$.

In 1988 the above problem was again proposed by Jordan [12]. The more general form, due to Klin [3, Problem BCC15.12], asks whether every transitive 2-closed permutation group contains a fixed-point-free element of prime order. A graph admitting such an automorphism is called a *polycirculant graph*, so we refer to the conjecture that no 2-closed transitive group is elusive as the *polycirculant conjecture*. Papers in the bibliography give results on the conjecture (for example, [19, 20]).

2 Examples

The first construction of elusive groups was given by Fein, Kantor and Schacher [9], and was as follows. Let p be a Mersenne prime $2^q - 1$, let G be the group

$$\mathrm{AGL}(1,p^2) = \{x \mapsto ax + b \mid a \in \mathrm{GF}(p^2) \setminus \{0\}, b \in \mathrm{GF}(p^2)\}$$

of affine transformations of $\mathrm{GF}(p^2)$, and let H be the subgroup $\mathrm{AGL}(1,p)$ consisting of these transformations where $a, b \in \mathrm{GF}(p)$. Then the left action of G on the set of left cosets of H gives rise to a transitive permutation group of degree $p(p+1)$ all of whose elements of prime order fix some point. For, a fixed-point-free element of prime order must have order dividing the degree, and here the only such primes are 2 and p; all elements of order 2 or p are conjugate in G, so all such elements lie in conjugates of H. Jones and Klin [11, 14] showed that this group is not 2-closed (cf. the end of Section 5 below), so it is not a counterexample to the polycirculant conjecture.

There is another way to view this example. The group G is a subgroup of $\mathrm{AGL}(2,p)$, and its action is on the set of lines of the affine plane. Clearly any involution or any translation fixes a line. So any subgroup of $\mathrm{AGL}(2,p)$ that is line-transitive and whose Sylow p-subgroup is the translation group of $\mathrm{AGL}(2,p)$ will be elusive.

In particular, if p is a Mersenne prime then any 2-transitive subgroup of $\mathrm{AGL}(2,p)$ whose order is not divisible by p^3 is elusive in its action on the lines of the affine plane. Among these groups we find all the sharply 2-transitive groups of degree p^2. These are the 1-dimensional affine groups over nearfields of order p^2. The nearfields were all determined by Zassenhaus [25]. In addition to the finite field $\mathrm{GF}(p^2)$, we can use a Dickson nearfield, or the exceptional nearfield of order 7^2. The affine groups over the Galois field and over the Dickson nearfield of order p^2 are normal subgroups of index 2 in the group $\mathrm{A\Gamma L}(1,p^2)$, which is also elusive.

For $p = 3$, all these examples are contained in the group M_{11}, in its 3-transitive action of degree 12 with point-stabilizer $\mathrm{PSL}(2,11)$. (The fact that this group is elusive follows from the fact that M_{11} has just one conjugacy class of elements of order 2 or 3; cf. [6].) The group M_{11} contains a further nonsolvable elusive subgroup, namely the transitive subgroup M_{10} (the two-point stabiliser in M_{12} in its usual 5-transitive action), with point-stabilizer A_5. All elusive groups of degree 12 are conjugate to subgroups of M_{11}, and together with their inclusions are shown in the diagram below.

$$\begin{tikzcd}[column sep=small]
 & & M_{11} \arrow[dl, dash] \arrow[dr, dash] & \\
 & \mathrm{A\Gamma L}(1,9) \arrow[dl, dash] \arrow[dr, dash] & & M_{10} = A_6 \cdot 2 \arrow[dl, dash] \\
\mathrm{AGL}(1,9) = 3^2 : C_8 & & M_9 = 3^2 : Q_8 &
\end{tikzcd}$$

We note here that the group in Example 2.3 of [21] of degree 12 is in fact not elusive, since it contains a Sylow 3-subgroup of $\mathrm{AGL}(2,3)$.

Jones and Klin [11, 14] showed that some of these examples of degree 12 can be used to build examples of degree $2^a \cdot 3$ for any $a \geq 2$. We generalise this, replacing 3 with any Mersenne prime p in Theorem 3.3, and also replacing the normal subgroup $C_p \times C_p$ by $C_{p^m} \times C_{p^m}$ in Theorem 3.2.

Finally, here are four more elusive groups which do not obviously belong to infinite families.

(a) M_{11} and M_{10} acting on 12 points;

(b) a group with structure

$$7^3 : (3^{1+2} : Q_8) \quad \text{or} \quad 7^3 : (3^{1+2} : \mathrm{SL}(2,3))$$

(a subgroup of $\mathrm{AGL}(3,7)$ whose linear group has a normal extraspecial subgroup – this group belongs to the Aschbacher class $\mathcal{C}_6$, see [1]), with point-stabilizer $7^2 : (3^2 : 2)$ or $7^2 : (3^2 : 6)$, respectively, acting on 84 points.

The elusiveness of (b) was verified by a computer calculation using MAGMA in order to check the conditions of Theorem 3.1.

We note that Giudici [10] has determined all the quasiprimitive elusive permutation groups, and shown that none of them is 2-closed. (A permutation group is *quasiprimitive* if every non-trivial normal subgroup is transitive.) In fact his result is more general; it is only necessary to assume that one minimal normal subgroup is transitive.

3 Affine constructions

The basic construction of elusive affine groups can be stated as follows.

Theorem 3.1 *Let G be a subgroup of $\mathrm{GL}(V)$ for some finite vector space V, and suppose that G has order prime to the characteristic of V. Let H be a subgroup of G, and W an H-invariant proper subspace of V. Then the action of $V : G$ on the cosets of $W : H$ is elusive if and only if*

(a) the translates of W by G cover V; and

(b) every conjugacy class of elements of prime order in G meets H.

Proof This is clear from the fact that elements of prime order in $V : G$ lie either in V or in a conjugate of G. $\square$

Note that the degree of this elusive group is $|V : W| \cdot |G : H|$.

The next two results build new examples from old ones. The first is quite general.

Theorem 3.2 *Suppose that V, G, W, and H satisfy the hypotheses of Theorem 3.1, where V is a vector space over $\mathrm{GF}(q)$, $q = p^e$, and $\dim(V) = n$. Then, for any $m \geq 1$, there is an elusive group $G_m = (C_{p^m})^{en} : G$, having degree $|V : W|^m \cdot |G : H|$.*

Proof By the idempotent-lifting results in [7], Section 77, there is an integral representation of G on F^n, where F is a p-adic number field whose residue field is $\mathrm{GF}(q)$. Taking this representation modulo p^m (that is, modulo the mth power of the maximal ideal), we find a representation of G on a homocyclic abelian group $A = A_m$ of exponent p^m and rank en. Now multiplication by p^{m-1} induces a G-homomorphism from A to $p^{m-1}A$ (the socle of A) with kernel pA; so $p^{m-1}A$ is isomorphic to A/pA as a G-module. Note that the socle contains all elements of prime order in A. Finally, we may choose a subgroup B of A which projects onto W and intersects the socle of A in the image of W. Now $A : G$ acting on the cosets of $B : H$ is elusive. $\square$

This construction produces elusive groups of degree $p^m(p+1)$ for all Mersenne primes p and integers $m \geq 1$. Also from examples (b) at the end of Section 2, we obtain elusive groups of degree $7^m \cdot 12$ for all $m \geq 1$.

The next construction is more specific. Let G be an affine group $E : A$, where E is the additive group of $\mathrm{GF}(p^n)$ and A is a subgroup of order a in its multiplicative group. Let H be a subgroup $F : B$ of G, where F is a maximal subgroup of E and B is a subgroup of order b in the multiplicative group of $\mathrm{GF}(p)$. Assume that

(a) G acting on the cosets of H is elusive (equivalently, the images of F under A cover E, and every prime divisor of a divides b);

(b) $p^n \equiv 1 \bmod 4$, and the 2-part of $p^n - 1$ divides a (so b is even).

Now let A' be the unique subgroup of order $2a$ in the multiplicative group of $\mathrm{GF}(p^{2n})$. By (b), the additive group E' of $\mathrm{GF}(p^{2n})$ is the direct sum $E \oplus E^*$ of two A-invariant subspaces E, E^* interchanged by A'. Let $G' = E' : A'$ and $H' = (F \oplus E^*) : B$. Then we have:

Theorem 3.3 *Conditions (a) and (b) of the above construction hold for G' and H', with $2n$ and $2a$ replacing n and a.*

Proof Note that the 2-part of $p^{2n} - 1 = (p^n - 1)(p^n + 1)$ is twice the 2-part of $p^n - 1$, by (b). Now the assertions are clear. $\square$

In particular, G' acting on the cosets of H' is elusive, and its degree is twice that of G on the cosets of H. Since the same conditions hold, the construction can be continued indefinitely. Now, starting with the affine examples of degree $p(p+1)$, where p is a Mersenne prime, we obtain elusive groups of degree $p \cdot 2^n$ for any n such that $2^n > p$. Finally, we can then apply the construction of Theorem 3.2 to these examples, to obtain elusive groups of degree $p^m \cdot 2^n$ for all $m \geq 1$ (with p and n as before).

A similar recursive construction can be developed with an arbitrary prime in place of 2. However, we do not know of any starting groups for such a construction!

Further affine examples have recently been found by Giudici; details will appear in his thesis. No new degrees are given by these constructions.

4 Product constructions

We first show that certain standard group-theoretic constructions preserve the property of being elusive. If (G_1, Ω_1) and (G_2, Ω_2) are permutation groups, then the direct product $G_1 \times G_2$ has natural actions on the disjoint union $\Omega_1 \cup \Omega_2$ (the *intransitive action*) and on $\Omega_1 \times \Omega_2$ (the *product action*). Also, the permutational wreath product $G_1 \wr G_2$ has natural actions on $\Omega_1 \times \Omega_2$ (the *imprimitive action*) and on $\Omega_1^{\Omega_2}$ (the *product action*).

Theorem 4.1 (a) *If (G, Ω) is elusive and B is a system of blocks of imprimitivity such that G acts faithfully on B, then (G, B) is elusive.*

(b) *If (G, Ω) is elusive and H is a transitive subgroup of G, then (H, Ω) is elusive.*

(c) *If (G_1, Ω_1) and (G_2, Ω_2) are elusive, then $(G_1 \wr G_2, \Omega_1 \times \Omega_2)$ and $(G_1 \times G_2, \Omega_1 \times \Omega_2)$ are elusive.*

(d) *If (G, Ω) is elusive, $|G|$ is odd, and $|\Omega| = n$, then $(2^{n-1} : G, 2 \times \Omega)$ is elusive.*

(e) *If (G, Ω) is elusive then $(G \wr S_n, \Omega^n)$ is elusive for any $n \geq 2$.*

(f) *If $(H, H : K)$ is elusive and G is a non-split extension of H of prime index, then $(G, G : K)$ is elusive.*

Proof (a) An element of prime order that fixes a point fixes the block containing it.

(b) Obvious.

(c) An element of prime order in $G_1 \wr G_2$ induces either the identity or an element of prime order in G_2, and hence fixes an element of Ω_2. Then it induces an element of prime order in the corresponding copy of G_1, and so fixes a point of $\Omega_1 \times \Omega_2$. Since $G_1 \times G_2$ is a transitive subgroup of $G_1 \wr G_2$, the second part of (c) now follows from (b).

(d) This needs a little explaining. In ATLAS [6] notation, 2 denotes the cyclic group of order 2 acting regularly; and $2^{n-1} : G$ denotes the subgroup of $2 \wr G$ which is the semidirect product of the subgroup of the base group 2^n consisting of elements with an even number of non-trivial coordinates by the top group. Now an element of odd prime order fixes a block (since G is elusive), and hence a point. An element of order 2 is in the base group, and is trivial in some coordinate since n is odd. (It must be said that this part is somewhat speculative, since there are no known elusive groups of odd order.)

(e) Take an element g of prime order p in $G \wr S_n$. Then g induces an element $\overline{g} \in S_n$, with, say, a cycles of length p and b fixed points, where $ap + b = n$. Then $g \in (G \wr C_p)^a \times G^b$. By (c) and induction, it suffices to prove the result for $G \wr C_p$. We may assume that $\overline{g} = (1\ 2\ \dots\ p)$. In the imprimitive action of the wreath product, let $((\alpha_1, 1)\ (\alpha_2, 2)\ \dots\ (\alpha_p, p))$ be a cycle of g. Then the element $(\alpha_1, \alpha_2, \dots, \alpha_p) \in \Omega^p$ is fixed by g.

(f) Since the extension does not split, $G \setminus H$ contains no elements of prime order. $\square$

By (b) and (c), if G is a direct product of Fein–Kantor–Schacher examples (using the same or different Mersenne primes), then any subgroup that contains both the socle and a Sylow 2-subgroup of G is also elusive.

5 2-closure

We now show that we do not get any counterexamples to the polycirculant conjecture from any of these constructions. We denote the 2-closure of G by $G^{(2)}$: this is the group of all permutations preserving the G-orbits on Ω^2. We refer to Wielandt [24] for a discussion of 2-closure, and for some of the results used here. A 2-*orbit* of a permutation group G on Ω is an orbit of G on Ω^2.

Theorem 5.1 *Let (G_1, Ω_1) and (G_2, Ω_2) be transitive permutation groups. Then*

$$(G_1 \times G_2)^{(2)} = G_1^{(2)} \times G_2^{(2)} \quad \textit{and} \quad (G_1 \wr G_2)^{(2)} = G_1^{(2)} \wr G_2^{(2)}.$$

Hence the following are equivalent:

(a) *(G_1, Ω_1) and (G_2, Ω_2) are 2-closed;*

(b) *$(G_1 \times G_2, \Omega_1 \times \Omega_2)$ is 2-closed;*

(c) *$(G_1 \wr G_2, \Omega_1 \times \Omega_2)$ is 2-closed.*

Proof The 2-orbits of $G_1 \times G_2$ are of four types:

- $\{((\alpha_1,\alpha_2),(\alpha_1,\alpha_2)) : \alpha_1 \in \Omega_1, \alpha_2 \in \Omega_2\}$;
- $\{((\alpha_1,\alpha_2),(\beta_1,\alpha_2)) : (\alpha_1,\beta_1) \in O_1, \alpha_2 \in \Omega_2\}$, for each 2-orbit O_1 of G_1;
- $\{((\alpha_1,\alpha_2),(\alpha_1,\beta_2)) : \alpha_1 \in \Omega_1, (\alpha_2,\beta_2) \in O_2\}$, for each 2-orbit O_2 of G_2;
- $\{((\alpha_1,\alpha_2),(\beta_1,\beta_2)) : (\alpha_1,\beta_1) \in O_1, (\alpha_2,\beta_2) \in O_2\}$, for any 2-orbits O_1 of G_1 and O_2 of G_2.

From this description it is clear that, if $g_1 \in G_1^{(2)}$ and $g_2 \in G_2^{(2)}$, then $(g_1,g_2) \in (G_1 \times G_2)^{(2)}$. For the converse, note that any permutation in $(G_1 \times G_2)^{(2)}$ fixes the two natural block systems (the transitive closures of the sets of orbits of the second and third types above), so it is of the form (g_1,g_2), where g_1 and g_2 are permutations of Ω_1 and Ω_2 respectively; and then it follows that $g_1 \in G_1^{(2)}$ and $g_2 \in G_2^{(2)}$.

The orbits of $G_1 \wr G_2$ on pairs are of three types:

- $\{((\alpha_1,\alpha_2),(\alpha_1,\alpha_2)) : \alpha_1 \in \Omega_1, \alpha_2 \in \Omega_2\}$;
- $\{((\alpha_1,\alpha_2),(\beta_1,\alpha_2)) : (\alpha_1,\beta_1) \in O_1, \alpha_2 \in \Omega_2\}$, for each 2-orbit O_1 of G_1;
- $\{((\alpha_1,\alpha_2),(\beta_1,\beta_2)) : \alpha_1,\beta_1 \in \Omega_1, (\alpha_2,\beta_2) \in O_2\}$, for each 2-orbit O_2 of G_2.

Now the proof proceeds as before. $\square$

Theorem 5.2 *In the product action on $\Omega_2^{\Omega_1}$, we have*

$$(G_1 \wr G_2)^{(2)} \geq G_1^{(2)} \wr G_2.$$

Hence, if G_1 is not 2-closed, then neither is $G_1 \wr G_2$ in the product action.

Proof This is proved by Praeger and Saxl [23] under the hypothesis (which is not required for their proof) that G_2 is transitive. $\square$

To show that Theorems 3.1–3.3 do not provide counterexamples to the polycirculant conjecture, we need the following result. Recall that a *half-transitive* group has all its orbits of the same length. The following result is a slight generalisation of Wielandt [24], Lemma 8.4.

Theorem 5.3 *Let E be a half-transitive permutation group with orbits $\mathcal{O}_i$ having the properties*

(a) the action of E on each $\mathcal{O}_i$ is regular;

(b) if $\alpha_i \in \mathcal{O}_i$ and $\alpha_j \in \mathcal{O}_j$ for $i \neq j$, then either $E_{\alpha_i} = E_{\alpha_j}$ or $E_{\alpha_i}E_{\alpha_j} = E$.

Then $E^{(2)}$ contains a semiregular permutation.

Proof Define an equivalence relation $\sim$ on the set of orbits by $\mathcal{O}_i \sim \mathcal{O}_j$ if and only if $E_{\alpha_i} = E_{\alpha_j}$. Then E induces a semiregular group $E^{\mathcal{C}}$ on the union $\mathcal{C}$ of each equivalence class of orbits. Inequivalent orbits $\mathcal{O}_i$ and $\mathcal{O}_j$ have distinct kernels K_i and K_j satisfying $K_iK_j = E$. Thus the kernel of E on $\mathcal{C}$ acts transitively on all orbits outside $\mathcal{C}$. It follows from Wielandt's Dissection Theorem ([24], Theorem 6.5) that any orbit of E on pairs is invariant under the groups $E^{\mathcal{C}}$; so their direct product, which contains semiregular elements, is contained in $E^{(2)}$. $\square$

Note that the hypotheses hold if E is the Sylow p-subgroup of a group all of whose orbits have length p. (This special case is particularly easy to prove.)

Theorem 5.4 *Let G be one of the groups produced by Theorems 3.1, 3.2 or 3.3. Assume that, in the case of Theorem 3.1 or 3.2, if $Wg \neq W$, then $W + Wg = V$. (In particular this holds if W has codimension 1 in V.) Then $G^{(2)}$ is not elusive.*

Proof Such a group G has an elementary abelian normal subgroup E whose orbits satisfy the hypotheses of Theorem 5.3. $\square$

Note that the condition that W has codimension 1 in V holds in all our examples. Also note that sporadic examples (b), together with their lifts obtained using Theorem 3.2, are covered by the preceding result.

The 2-closures of the Fein-Kantor-Schacher groups $G = \mathrm{AGL}(1, p^2)$ are determined in [11]. The cosets of $H = \mathrm{AGL}(1, p)$ can be identified with the $p(p+1)$ lines in the affine plane $\mathrm{GF}(p^2)$ over $\mathrm{GF}(p)$, so that H fixes the line $\mathrm{GF}(p) \subset \mathrm{GF}(p^2)$. The orbits of G on pairs are the diagonal $\{(\alpha, \alpha)\}$, the set

$$\{(\alpha, \beta) \mid \alpha \parallel \beta,\ \alpha \neq \beta\}$$

of distinct parallel pairs, and p sets of pairs satisfying $\alpha \parallel \beta^g$ where a fixed $g \in \mathrm{GF}(p^2) \setminus \mathrm{GF}(p)$ acts by multiplication, one set for each nontrivial coset of $\mathrm{GF}(p) \setminus \{0\}$ in $\mathrm{GF}(p^2) \setminus \{0\}$. The binary relations of this last form are all compositions of a single one of them, for which g is a generator of $\mathrm{GF}(p^2) \setminus \{0\}$. Since the automorphism group $S_p \wr C_{p+1}$ of this relation preserves all the other G-invariant relations, this is $G^{(2)}$.

Finally:

Theorem 5.5 *The sporadic examples M_{11} and M_{10} are not 2-closed.*

Proof This is clear for the 3-transitive group M_{11}. In the case of M_{10}, the normal subgroup $S = A_6$ has two orbits of length 6. Moreover, S is the product $S = S_\alpha S_\beta$ of stabilizers of points α, β in different orbits [17]; equivalently, S_α is transitive on β^S. So by Wielandt's Dissection Theorem ([24], Theorem 6.5), the 2-closure of G contains $A_6 \times A_6$. $\square$

6 Open problems

We conclude with several open problems.

Problem 1 Prove the polycirculant conjecture, namely, that there is no elusive 2-closed permutation group.

Problem 2 Does the set of degrees of elusive groups have density zero? Theorem 4.1(c) shows that this set is multiplicatively closed; and, as noted, it contains $p^m \cdot 2^n$ for p a Mersenne prime, $m > 0$, and $2^n > p$, and also $7^m \cdot 12$ for $m > 0$.

Problem 3 As noted earlier, Marušič and Scapellato [21] proved that the automorphism group of a vertex-transitive cubic graph is not elusive. Cameron and Sheehan [4, Problem BCC17.12] conjectured that there is a function f, satisfying $f(n) \to \infty$ as $n \to \infty$, such that a vertex-transitive automorphism group of a vertex-transitive cubic graph on n vertices has a semiregular subgroup of order at least $f(n)$. Prove this conjecture.

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