On the automorphism group of the m-coloured random graph

Peter J. Cameron and Sam Tarzi
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London E1 4NS, UK
p.j.cameron@qmul.ac.uk

Abstract

Let R_m be the (unique) universal homogeneous m-edge-coloured countable complete graph ($m \geq 2$), and G_m its group of colour-preserving automorphisms. The group G_m was shown to be simple by John Truss. We examine the automorphism group of G_m, and show that it is the group of permutations of R_m which induce permutations on the colours, and hence an extension of G_m by the symmetric group of degree m. We show further that the extension splits if and only if m is odd, and in the case where m is even and not divisible by 8 we find the smallest supplement for G_m in its automorphism group.

1 Introduction

Fix an integer $m \geq 2$, and let R_m be the unique homogeneous universal m-edge-colouring of the countable complete graph (see Truss [6]). (Universality means that any m-edge-coloured finite or countable complete graph is embeddable in R_m, and homogeneity means that every colour-preserving isomorphism between finite subgraphs extends to an automorphism of R_m. The uniqueness is a special case of Fraïssé's theory of countable homogeneous structures. The graph R_m is the 'random m-edge-coloured complete graph':

that is, we colour edges independently at random, we obtain R_m with probability 1. More relevant to us is the fact that the isomorphism class of R_m is residual in the set of all m-coloured complete graphs on a fixed countable vertex set. See [1] for discussion.)

Let $\mathrm{Aut}(R_m)$ be the group of permutations of the vertex set fixing all the colours. Truss [6] showed that $\mathrm{Aut}(R_m)$ is a simple group.

For any permutation π of the set of colours, let R_m^π be the graph obtained by applying π to the colours. Then R_m^π is universal and homogeneous, and hence isomorphic to R_m. This means that, if $\mathrm{Aut}^*(R_m)$ is the group of permutations of the vertex set which induce permutations of the colours, then $\mathrm{Aut}^*(R_m)$ induces the symmetric group $\mathrm{Sym}(m)$ on the colours; so $\mathrm{Aut}^*(R_m)$ is an extension of $\mathrm{Aut}(R_m)$ by $\mathrm{Sym}(m)$.

The first question we consider here is: when does this extension split? That is, when is there a complement for $\mathrm{Aut}(R_m)$ in $\mathrm{Aut}^*(R_m)$ (a subgroup of $\mathrm{Aut}^*(R_m)$ isomorphic to $\mathrm{Sym}(m)$ which permutes the colours)? We also show that $\mathrm{Aut}^*(R_m)$ is the automorphism group of the simple group $\mathrm{Aut}(R_m)$ (so that the outer automorphism group of this group is $\mathrm{Sym}(m)$).

Theorem 1 *The group* $\mathrm{Aut}^*(R_m)$ *splits over* $\mathrm{Aut}(R_m)$ *if and only if* m *is odd.*

Theorem 2 *The automorphism group of* $\mathrm{Aut}(R_m)$ *is* $\mathrm{Aut}^*(R_m)$.

2 Proof of Theorem 1

We show first that the extension does not split if m is even. Suppose that a complement exists, and let s be an element of this complement acting as $(1,2)(3,4)\cdots(m-1,m)$ on the colours. Then s maps the subgraph with colours $1,3,\ldots,m-1$ to its complement. But this is impossible, since the edge joining points in a 2-cycle of s has its colour fixed.

Now suppose that m is odd; we are going to construct a complement.

First, we show that there exists a function f from pairs of distinct elements of $\mathrm{Sym}(m)$ to $\{1,\ldots,m\}$ satisfying

- $f(x,y) = f(y,x)$ for all $x \neq y$;
- $f(xg,yg) = f(x,y)^g$ for all $x \neq y$ and all g.

To do this, we first define $f(1,y)$ for $y \neq 1$ arbitrarily subject to the condition $f(1,x^{-1}) = f(1,x)^{x^{-1}}$. Note that this condition requires $f(1,s)^s = f(1,s)$ whenever s is an involution; but this is possible, since any involution has a fixed point (as m is odd). Then we extend to all pairs by defining $f(x,y) = f(1,yx^{-1})^x$. A little thought shows that no conflict arises.

Now we take a countable set of vertices, and let $\mathrm{Sym}(m)$ act semiregularly on it. Each orbit is naturally identified with $\mathrm{Sym}(m)$; we let x_i denote the element identified with x in the ith orbit, as $i \in \mathbb{N}$ (where orbits are indexed by natural numbers). Then we colour the edges within each orbit by giving $\{x_i, y_i\}$ the colour $f(x,y)$. For edges between orbits i and j, with $i < j$, we colour $\{x_i, 1_j\}$ arbitrarily, and then give $\{y_i, z_k\}$ the image of the colour of $\{(yz^{-1})_i, 1_j\}$ under z.

Clearly the group $\mathrm{Sym}(m)$ permutes the colours of the edges consistently, the same way as it permutes $\{1,\ldots,m\}$.

Next we show that a residual set of the coloured graphs we obtain are isomorphic to R_m. We have to show that, given m finite disjoint sets of vertices, say $U_1,\ldots,U_m$, the set of graphs containing a vertex v joined by edges of colour i to all vertices in U_i (for $i = 1,\ldots,m$) is open and dense. The openness is clear. To see that it is dense, note that the m finite sets are contained in the union of a finite number of orbits (say those with index less than N); then, for any $i \geq N$, we are free to choose the colours of the edges joining these vertices to 1_i arbitrarily.

Now by construction, the group $\mathrm{Sym}(m)$ we have constructed meets $\mathrm{Aut}(R_m)$ in the identity; so it is the required complement.

How close can we get when m is even? The construction in the second part can easily be modified to show that, if there is a group G which acts as $\mathrm{Sym}(m)$ on the set $\{1,\ldots,m\}$, in such a way that all involutions in G have fixed points on $\{1,\ldots,m\}$, then G is a supplement for $\mathrm{Aut}(R_m)$ in $\mathrm{Aut}^*(R_m)$ (that is, $G.\,\mathrm{Aut}(R_m) = \mathrm{Aut}^*(R_m)$), and $G \cap \mathrm{Aut}(R_m)$ is the kernel of the action of G on $\{1,\ldots,m\}$. We simply replace $\mathrm{Sym}(m)$ by G in the construction, and in place of $f(xg,yg) = f(x,y)^g$ we require that $f(xg,yg) = f(x,y)^{g\phi}$, where ϕ is the action of G on $\{1,\ldots,m\}$.

If m is even but not a multiple of 8, then there is a double cover of $\mathrm{Sym}(m)$, for m even, in which the fixed-point-free involutions lift to elements of order 4. (There are two double covers of $\mathrm{Sym}(n)$ for $n \geq 4$, described in [4, Chapter 2] and called there $\tilde{S}_m$ and $\hat{S}_m$. In $\tilde{S}_m$, the product of r disjoint transpositions lifts to an element of order 4 if and only if $r \equiv 1$ or $2 \bmod 4$, while in $\hat{S}_m$, the condition is that $r \equiv 2$ or 3 mod 4.) This shows that there is a supplement meeting $\mathrm{Aut}(R_m)$ in a group of order 2 for m even but not divisible by 8.

What happens in the remaining case, when m is a multiple of 8? Is there a finite supplement, and what is the smallest such?

3 Proof of Theorem 2

Since $\mathrm{Aut}(R_m)$ is primitive and not regular, its centraliser in the symmetric group is trivial; so $\mathrm{Aut}^*(R_m)$ acts faithfully on $\mathrm{Aut}(R_m)$ by conjugation. We have to show that there are no further automorphisms.

A permutation group G of countable degree is said to have the *small index property* if any subgroup H satisfying $|G : H| < 2^{\aleph_0}$ contains the pointwise stabiliser of a finite set; it has the *strong small index property* if if any subgroup H satisfying $|G : H| < 2^{\aleph_0}$ lies between the pointwise and setwise stabiliser of a finite set.

Step 1 R_m has the strong small index property.

This is proved by a simple modification of the arguments for the case $m = 2$. The small index property is proved by Hodges *et al.* [3], using a result of Hrushovski [5]; the strong version is a simple extension due to Cameron [2].

Hrushovski showed that any finite graph X can be embedded into a finite graph Z such that all isomorphisms between subgraphs of X extend to automorphisms of Z. Moreover, the graph Z is vertex-, edge- and nonedge-transitive. He uses this to construct a generic countable sequence of automorphisms of R. To extend this to R_m is comparatively straightforward. It is necessary to work with $(m-1)$-edge-coloured graphs (regarding the mth colour as 'transparent'). Now the arguments of Hodges *et al.* and Cameron go through essentially unchanged.

Step 2 Since $\mathrm{Aut}(R_m)$ acts primitively on the vertex set, with permutation rank $m+1$, the vertex stabilisers are maximal subgroups of countable index with $m+1$ double cosets. Moreover, any further subgroup of countable index has more than $m+1$ double cosets.

For let H be a maximal subgroup of countable index. By the strong SIP, H is the stabiliser of a k-set X. If g maps X to a disjoint k-set, then HgH

determines the colours of the edges between X and X^g, up to permutations of these two sets. By universality, there are at least $m^{k^2}/(k!)^2$ such double cosets. Now it is not hard to prove that $m^{k^2}/(k!)^2 > m$ for $k \geq 2$. Hence we must have $k = 1$.

Step 3 It follows that any automorphism permutes the vertex stabilisers among themselves, so is induced by a permutation of the vertices which normalises $\mathrm{Aut}(R_m)$. To finish the proof, we show that the normaliser of $\mathrm{Aut}(R_m)$ in the symmetric group is $\mathrm{Aut}^*(R_m)$.

This is straightforward. A vertex permutation which normalises $\mathrm{Aut}(R_m)$ must permute among themselves the $\mathrm{Aut}(R_m)$-orbits on pairs of vertices, that is, the colour classes; so it belongs to $\mathrm{Aut}^*(R_m)$.

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