Counting false entries in truth tables of bracketed formulae connected by implication

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Abstract

In this note we count the number of rows f_n with the value "false" in the truth tables of all bracketed formulae with n variables connected by the binary connective of implication. We find a recurrence and an asymptotic formulae for f_n. We also show that the ratio of false rows to the total number of rows converges to the constant $(3-\sqrt{3})/6$.

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1 Introduction

In this paper we study enumerative and asymptotic questions on formulae of propositional calculus which are correctly bracketed chains of implications.

For brevity, we represent truth values of propositional variables and formulae by 1 (for "true") and 0 (for "false"). A propositional function of n

variables $p_1, \ldots, p_n$ is simply a function from $\{0,1\}^n$ to $\{0,1\}$. As is well known, there are 2^{2^n} propositional functions, each of which can be represented by a formula involving the connectives $\neg$, $\vee$ and $\wedge$. A valuation is an assignment of values to the variables $p_1, \ldots, p_n$, with consequent assignment of values to formulae.

The function represented by a formula is conveniently calculated using a truth table. Each row of the truth table corresponds to a valuation.

We are interested in *bracketed implications*, which are formulae obtained from $p_1 \to p_2 \to \cdots \to p_n$ by inserting brackets so that the result is well-formed; the binary connective $\to$ ("implies") is defined as usual by the rule that, for any valuation ν,

$$\nu(\phi \to \psi) = \begin{cases} 0 & \text{if } \nu(\phi) = 1 \text{ and } \nu(\psi) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Here, for example, are the truth tables for the two bracketed implications in $n = 3$ variables.

p_1	p_2	p_3	$p_1 \to (p_2 \to p_3)$	$(p_1 \to p_2) \to p_3$
1	1	1	1	1
1	1	0	0	0
1	0	1	1	1
1	0	0	1	1
0	1	1	1	1
0	1	0	1	0
0	0	1	1	1
0	0	0	1	0

Our concern is with the set of propositional functions defined by bracketed implications. The following uniqueness lemma shows that it suffices to work with the formulae. Two formulae are logically equivalent if they define the same propositional function.

Lemma 1.1 *Two bracketed implications are logically equivalent if and only if they are equal.*

Proof We show how to recover the bracketing from the propositional function defined by such a formula. Our proof is by induction on n, the result being clear for $n \leq 2$.

Let ϕ be a bracketed implication. Let valuations ν_i and $\nu_{i,j}$ be defined by

$$\begin{aligned} \nu_i(p_j) &= 0 \text{ if } j = i, \quad 1 \text{ otherwise;} \\ \nu_{i,j}(p_k) &= 0 \text{ if } k = i \text{ or } k = j, \quad 1 \text{ otherwise.} \end{aligned}$$

Now it is straightforward to check that $\nu_n(\phi) = 0$, while $\nu_i(\phi) = 1$ for $i \neq n$.

Suppose that ϕ has the form $\psi \to \chi$, where ψ and χ are bracketed implications involving $p_1, \ldots, p_r$ and $p_{r+1}, \ldots, p_n$ respectively. Then, for $i \leq r$, we have $\nu_{i,n}(\chi) = 0$, while $\nu_{i,n}(\psi) = 1$ if $i < r$, $\nu_{r,n}(\psi) = 0$. We conclude that $\nu_{i,n}(\phi) = 0$ if $i < r$ while $\nu_{r,n}(\phi) = 1$. Hence we can determine the value of r. By the induction hypothesis, the bracketings of ψ and χ are determined by the propositional function, and hence the bracketing of ϕ is determined. $\square$

We could also consider *permuted bracketed implications*, which are well-formed bracketings of $p_{i_1} \to p_{i_2} \to \cdots \to p_{i_n}$, where $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$. Here the situation is less satisfactory; we can count formulae, but the analogue of our uniqueness lemma doess not hold (for example, $p_1 \to (p_2 \to p_3)$ and $p_2 \to (p_1 \to p_3)$ define the same propositional function), and we do not know how to count propositional functions represented by permuted bracketed implications, or the rows with value "false" in the corresponding truth tables.

2 The number of false rows

It is well known that the number of bracketings of a product of n terms is the Catalan number

$$C_n = \frac{1}{n}\binom{2n-2}{n-1},$$

whose generating function is

$$\sum_{n \geq 1} C_n x^n = (1 - \sqrt{1-4x})/2$$

(see [2, page 61]). Then C_n is the number of bracketed implications in n propositional variables, and by the uniqueness lemma of the preceding section, it is also the number of propositional functions or truth tables defined by such formulae.

Let f_n be number of rows with the value "false" in the truth tables of all bracketed implications.

Proposition 2.1

$$f_n = \sum_{i=1}^{n-1} (2^i C_i - f_i) f_{n-i}.$$

Proof A false row comes from an expression $\psi \to \chi$ where $\nu(\phi) = 1$ and $\nu(\psi) = 0$. If ψ contains i variables, then χ contains $n - i$, and the number of choices is given by the summand in the proposition. $\square$

Using this Proposition, it is straightforward to calculate values of f_n for small n. The first 22 values are

$$\begin{aligned}(f_n)_1^\infty &= 1, 1, 4, 19, 104, 614, 3816, 24595, 162896, 1101922, 7580904,\\ &\quad 52878654, 373100272, 2658188524, 19096607120, 138182654595,\\ &\quad 1006202473888, 7367648586954, 54214472633064,\\ &\quad 400698865376842, 2973344993337520, 22142778865313364, \ldots\end{aligned}$$

Let $F(x)$ be the generating function for f_n, that is, $F(x) := \sum f_n x^n$. Also let $G(x) := \sum 2^n C_n x^n$ be the generating function for the total number of rows. Then Proposition 2.1 gives

$$F(x) = x + F(x)(G(x) - F(x)),$$

where $G(x)$ can be obtained from the generating function of C_n by replacing x by $2x$: that is,

$$G(x) = (1 - \sqrt{1 - 8x})/2.$$

Substituting the result into the equation for $G(x)$ gives the quadratic equation

$$2F(x)^2 + F(x)\left(1 + \sqrt{1 - 8x}\right) - 2x = 0.$$

This can be solved, to give the following:

Proposition 2.2 *The generating function for the sequence* (f_n) *is given by*

$$F(x) = \frac{-1 - \sqrt{1 - 8x} + \sqrt{2 + 2\sqrt{1 - 8x} + 8x}}{4}.$$

(As with the Catalan numbers, the choice of sign in the square root is made to ensure that $F(0) = 0$.) With the help of Maple we can obtain the first 22 terms of the above series, and hence give the first 22 values of f_n; these agree with the values found from the recurrence relation.

3 Asymptotic analysis

In this section we want to get an asymptotic formula for the coefficients of the generating function

$$F(x) = \frac{-1 - \sqrt{1-8x} + \sqrt{2 + 2\sqrt{1-8x} + 8x}}{4}$$

from Proposition 2.2. We use the following result [1, page 389]:

Proposition 3.1 *Let a_n be a sequence whose terms are positive for sufficiently large n. Suppose that $A(x) = \sum_{n\geq 0} a_n x^n$ converges for some value of $x > 0$. Let $f(x) = (-\ln(1 - x/r))^b (1 - x/r)^c$, where c is not a positive integer, and we do not have $b = c = 0$. Suppose that $A(x)$ and $f(x)$ each have a singularity at $x = r$ and that $A(x)$ has no singularities in the interval $[-r, r)$. Suppose further that $\lim_{x\to r} \frac{A(x)}{f(x)}$ exists and has nonzero value L. Then*

$$a_n \sim \begin{cases} L\binom{n-c-1}{n}(\ln n)^b r^{-n}, & \textit{if } c \neq 0, \\ \frac{Lb(\ln n)^{b-1}}{n}, & \textit{if } c = 0. \end{cases}$$

Note We also have

$$\binom{n-c-1}{n} \sim \frac{n^{-c-1}}{\Gamma(-c)},$$

where the gamma-function

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t}\,\mathrm{d}t$$

satisfies $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1/2) = \sqrt{\pi}$. It follows that $\Gamma(-1/2) = -\sqrt{\pi}/2$.

Back to our problem, recall that $G(x) = (1 - \sqrt{1-8x})/2$, therefore

$$F(x) = \frac{(G(x) - 1) + \sqrt{(1 - G(x))^2 + 4x}}{2}.$$

To study $F(x)$, we begin with the simple $G(x)$. This $G(x)$ could easily be studied by using the explicit formula for its coefficients, which is $2^n\binom{2n-2}{n-1}/n$. But our aim is to understand how to handle the square root singularity.

The square root has a singularity at $1/8$. We have $G(1/8) = 1/2$, so we would not be able to divide $G(x)$ by a suitable $f(x)$ as required in proposition

1. To create a function which vanishes at $\frac{1}{8}$, we simply look at $A(x) = G(x) - 1/2$ instead. That is, let

$$f(x) = (1 - x/r)^{1/2} = (1 - 8x)^{1/2}.$$

Then

$$L = \lim_{x \to 1/8} \frac{A(x)}{\sqrt{1-8x}} = -\frac{1}{2}.$$

Now by using Proposition 1 and the following note,

$$g_n \sim -\frac{1}{2} \binom{n - \frac{3}{2}}{n} \left(\frac{1}{8}\right)^{-n} \sim -\frac{1}{2} \frac{8^n n^{-3/2}}{\Gamma(-1/2)} = \frac{2^{3n-2}}{\sqrt{\pi n^3}}.$$

We are now ready to tackle $F(x)$, and state the main theorem of the paper.

Theorem 3.2 *Let f_n be number of rows with the value false in the truth tables of all the bracketed implications with n variables. Then*

$$f_n \sim \left(\frac{3-\sqrt{3}}{6}\right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}.$$

Proof We have

$$F(x) = \frac{-1 - \sqrt{1-8x} + \sqrt{2 + 2\sqrt{1-8x} + 8x}}{4}.$$

We find that $r = \frac{1}{8}$, and $f(x) = \sqrt{1-8x}$. Since $F(1/8) = (-1+\sqrt{3})/4 \neq 0$, we use the idea from above, and we let $A(x) = F(x) - F(1/8)$.

$$\lim_{x \to 1/8} \frac{A(x)}{f(x)} = \lim_{x \to 1/8} \frac{-\sqrt{1-8x} + \sqrt{2 + 2\sqrt{1-8x} + 8x} - \sqrt{3}}{4\sqrt{1-8x}}.$$

Let $v = \sqrt{1-8x}$. Then

$$\begin{aligned} L &= \lim_{v \to 0} \frac{-v + \sqrt{(1+v)(3-v)} - \sqrt{3}}{4v} \\ &= \lim_{v \to 0} \frac{-v + \sqrt{3 + 2v - v^2} - \sqrt{3}}{4v} \\ &= \lim_{v \to 0} \frac{-1 + (1-v)(3 + 2v - v^2)^{-1/2}}{4} \\ &= -\frac{3-\sqrt{3}}{12}, \end{aligned}$$

where we have used l'Hôpital's Rule in the penultimate line.

Finally,

$$f_n \sim -\frac{3-\sqrt{3}}{12}\binom{n-\frac{3}{2}}{n}\left(\frac{1}{8}\right)^{-n} \sim \left(\frac{3-\sqrt{3}}{6}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}},$$

and the proof is finished. □

The importance of the constant $(3-\sqrt{3})/6 = 0.2113248654$ lies in the following fact:

Corollary 3.3 *Let g_n be the total number of rows in all truth tables for bracketed implications with n variables, and f_n the number of rows with the value "false". Then* $\lim_{n\to\infty} f_n/g_n = (3-\sqrt{3})/6$.

The table below illustrates the convergence.

n	f_n	g_n	f_n/g_n
1	1	2	0.5
2	2	4	0.25
3	4	16	0.25
4	19	80	0.2375
5	104	428	0.2321428571
6	614	2688	0.228422619
7	3816	16896	0.2258522727
8	424595	109824	0.2239492279
9	162896	732160	0.2224868881
10	1101922	4978688	0.2213277876

For $n = 100$ the ratio is 0.2122908650, and for $n = 1000$ it is 0.2114211279.

References

[1] E. A. Bender, *Foundations of Applied Combinatorics*, Addison-Wesley Publishing Company, 1991.

[2] P. J. Cameron, *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press, Cambridge, 1994.