Matrices with zero row and column sum

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In Kirchhoff's Matrix-Tree Theorem, it is shown that the Laplacian matrix of a graph (which is a symmetric real matrix with row and column sums zero) has the property that all its cofactors are equal, the common value being the number of spanning trees of the graph. In fact this is a much more general property of a matrix; symmetry and real numbers are not required. As a corollary, we obtain another formula for the number of spanning trees: it is the product of the non-principal eigenvalues of the Laplacian matrix divided by the number of vertices.

Proposition 1 *Let A be a square matrix over an arbitrary field with all row and column sums zero. Then all cofactors of A are equal.*

Proof Let A be $n \times n$, and let J be the all-1 matrix of size $n \times n$.

We evaluate $\det(A + J)$ by performing row and column operations. We distinguish the first row and column for simplicity, but the same applies to any row and column. We do the following:

- Add all other rows of $A + J$ to the first. Then every entry in the first row is n, while the other rows are unaffected.
- Add all other columns to the first. In the result, the $(1, 1)$ entry is n^2; all other entries of the first row and column are n; and the remaining entries are unaffected.
- Take out a factor n from the first row.
- Subtract the first row from each of the other rows. The $(1, 1)$ entry of the result is n; the remaining entries of the first column are zero; and the entries not in the first row or column are precisely those of A, since we subtract 1 from each of them.

Thus, $\det(A + J) = n^2 A_{11}$, where A_{11} is the $(1, 1)$ cofactor of A. As remarked, the same applies to any cofactor; this shows that all cofactors are equal.

Now suppose that A is real symmetric, with eigenvalues $\lambda_1 = 0$ (corresponding to the all-1 eigenvector), $\lambda_2, \ldots, \lambda_n$. Then A and J are simultaneously diagonalisable, so the eigenvalues of $A + J$ are n, $\lambda_2, \ldots, \lambda_n$. Their product is the determinant. So we have

$$A_{11} = (\lambda_2 \cdots \lambda_n)/n.$$