Problems from the DocCourse: Day 9

A problem held over from last week

1. Let U be a subspace of the space of $m \times n$ complex matrices which contains no matrix of rank 1. Prove that $\dim(U) \leq (m-1)(n-1)$.

Is this true for real matrices?

Hence [or assuming the result if you can't prove it] show that if

$$A = \bigoplus_{i \geq 0} V_i$$

is a graded algebra over $\mathbb{C}$ (that is, $V_0 = \mathbb{C} \cdot 1$ and $V_i \cdot V_j \subseteq V_{i+j}$) which is an integral domain (i.e. has no divisors of zero), then

$$\dim(V_{i+j}) \geq \dim(V_i) + \dim(V_j) - 1.$$

Deduce that, if G is an oligomorphic permutation group for which the graded algebra A^G is an integral domain, then

$$f_{i+j}(G) \geq f_i(G) + f_j(G) - 1.$$

Find an example where equality holds for all i and j.

Bases and Schreier–Sims algorithm

2. Show that the permutations $(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$ and $(1,2,3)(4,5,7)(8,9,11)$ generate a 2-transitive permutation group of degree 12. (Actually the group generated by these two permutations is the *Mathieu group* M_{12}, which is sharply 5-transitive: if you enjoy this kind of calculation, you may want to prove this.)

3. Let G be the group induced on 2-sets by the symmetric group on $\{1,2,\ldots,m\}$. Find the minimum base size and the size(s) of bases produced by the greedy algorithm for G.

4. Prove that if the permutation group G has irredundant bases of sizes m_1 and m_2, then it has irredundant bases of all possible sizes between m_1 and m_2.

Give an example to show that this is false for minimal bases.

Matroids

5. The *uniform matroid* $U(k,n)$ is defined as follows: $E = \{1,2,\ldots,n\}$, and the independent sets are all subsets of cardinality at most k.

- What are the bases, the circuits, and the hyperplanes, of $U(k,n)$?
- Show that if G is a permutation group on E with the property that the stabiliser of any k points is the identity but any $k-1$ points are fixed by some non-identity element, show that the irredundant bases for G are the bases of $U(k,n)$.
- [Harder!] Show that a group with the above property is $(k-1)$-transitive if $k > 1$. Hint: Use induction on k. The most difficult case is the starting case $k = 2$.

Notes: (a) Prove that, if a group with this property with $k = 2$ has degree n and is transitive, then it contains exactly $n-1$ derangements. *Frobenius' Theorem* asserts that these derangements, together with the identity, form a normal subgroup of G which acts regularly; you may use this if you wish, but it is possible to do without.

(b) All groups satisfying this property with $k \geq 3$ have been determined; it is not necessary to use the Classification of Finite Simple Groups in this classification. In particular, if $k \geq 5$, then such a group must be sharply k-transitive.