Problems from the DocCourse: Day 4

Diagonal groups

This exercise gives another, more computational, approach to diagonal groups.

The *diagonal group* $D(T,n)$, where T is a group and n a positive integer, is defined as the permutation group on the set

$$\Omega = T^n = \{[x_1,\ldots,x_n] : x_1,\ldots,x_n \in T\}$$

generated by the following permutations:

(a) the group T^n acting by right translation, that is, the permutations

$$[x_1,\ldots,x_n] \mapsto [x_1t_1,\ldots,x_nt_n]$$

for $t_1,\ldots,t_n \in T$;

(b) the automorphism group of T, acting coordinatewise, that is,

$$[x_1,\ldots,x_n] \mapsto [x_1^\alpha,\ldots,x_n^\alpha]$$

for $\alpha \in \mathrm{Aut}(T)$;

(c) the symmetric group S_n, acting by permuting the coordinates, that is,

$$\pi : [x_1,x_2,\ldots,x_n] \mapsto [x_{1\pi},x_{2\pi},\ldots,x_{n\pi}]$$

for $\pi \in S_n$;

(d) the permutation

$$\tau : [x_1,x_2,\ldots,x_n] \mapsto [x_1^{-1},x_1^{-1}x_2,\ldots,x_1^{-1}x_n].$$

Your job is to match this up with the description of this group which was given in the lectures, and which appears in the O'nan–Scott theorem.

The diagonal group is usually defined, in the case where T is simple, as the group

$$G = T^{n+1}(\mathrm{Out}(T) \times S_{n+1})$$

acting on the set of right cosets of the subgroup

$$H = \mathrm{Aut}(T) \times S_{n+1}$$

by right multiplication, where the inner automorphisms are identified with the diagonal subgroup

$$T^\sharp = \{(t,t,\ldots,t) : t \in T\}$$

of T^{n+1}.

Show that each right coset of H in G has a unique coset representative of the form $(1,t_1,t_2,\ldots,t_n)$, for $t_1,\ldots,t_n \in T$. Show further that the bijection

$$H(1,t_1,\ldots,t_n) \Leftrightarrow [t_1,\ldots,t_n]$$

is an isomorphism between the set $G:H$ of right cosets and the set T^n appearing in the first description. [Hint: The transposition of the first two coordinates translates into the permutation (d). How do we get right multiplication by an element $(t,1,\ldots,1)$?]

In the case $n = 1$, show that the diagonal group is obtained by taking $\Omega = T$ and the group generated by the left and right translations, automorphisms of T, and the map $t \mapsto t^{-1}$.

From the book

4.8: This asks you to take the diagonal group for $n = 2$, $T = A_5$ (the alternating group of degree 5), and compute the number of orbits of the stabiliser of a point.

4.23: This asks you to prove, assuming the truth of Schreier's conjecture, that the only finite 8-transitive groups are the symmetric and alternating groups. For hints, see the exercise.

A challenge

Mathieu observed that, for any n with $5 \leq n \leq 33$, there is a primitive group of degree n which is not the symmetric or alternating group. (The numbers $2,3,4$ are too small.) Your challenge: For how many of these values can you find such a group?

It seems likely that Mathieu thought that a primitive group of degree n other than S_n and A_n would exist for all or most values of n. Now, with the Classification of Finite Simple Groups, we know that the reverse is the case!