Problems from the DocCourse: Day 10

Matroids and codes

1. You are given 12 coins, one of which is known to be either lighter or heavier than all the others; you are also given a beam balance. Devise a scheme of three weighings which will identify the odd coin and determine if it is light or heavy; the coins weighed at each step should not depend on the results of previous weighings.

Show that the problem with 13 coins cannot be solved in three weighings.

What is the connection between this problem and error-correcting codes over $\mathbb{Z}_3 = \{-1, 0, +1\}$?

2. Prove that a code with length n and minimum distance d over an alphabet of q symbols contains at most q^{n-d+1} codewords. [This is known as the *Singleton bound*: a code meeting it is said to be *maximum distance separable*, or MDS.]

Show that, if an MDS code is linear, then the corresponding matroid is a uniform matroid $U(k,n)$, with $k = n-d+1$.

3. Let G be a finite graph with edge set E. Prove that the set of subsets of E which are forests (that is, contain no cycle) forms the set of independent sets of a matroid on E, called the *graphic* matroid $M(G)$. What is its rank?

4. The *chromatic polynomial* $P(G;x)$ of a graph G is the function whose value at a positive integer x is equal to the number of proper vertex-colourings of G with x colours.

- Prove that $P(G,x) = P(G\backslash e,x) - P(G/e,x)$ for any edge e, where $G\backslash e$ and G/e are the graphs obtained by deleting and contracting the edge e, respectively.

- Deduce that $P(G,x)$ is a polynomial in x whose degree is equal to the number of vertices of G (more precisely, there is such a polynomial which takes the value $P(G,x)$ at the positive integer x for all $x \in \mathbb{N}$).

- Let $M(G)$ be the graphic matroid associated with G. Prove that

$$P(G,x) = (-1)^{n-k} x^k T(M(G); 1-x, 0),$$

where n is the number of vertices of G and k the number of connected components.

5. (a) Show that the uniform matroid $U(2,n)$ is representable over a field of order q if and only if $n \leq q+1$. Show that, in the case $n = q+1$, the representation is unique [in a suitable sense].

(b) [Harder] Show that, if the uniform matroid $U(3,n)$ is representable over a field of order q, then

$$n \leq \begin{cases} q+1 & \text{if } q \text{ is odd,} \\ q+2 & \text{if } q \text{ is even.} \end{cases}$$

Construct representations attaining the bounds.

6. This is a question to which I don't know the answer. A prize is offered for a solution.

Which graphic matroids are associated with IBIS groups?