Problems from the DocCourse: Day 1

Standard problems

The following problems from the book *Permutation groups* were suggested, with "star ratings":

> 1.1^*, 1.2^{**}, 1.5^{**}, 1.7^{**}, 1.11^*, 1.14^*, 1.16^{**}, 1.19^{***}, 1.28^*, 1.34^{**}, 1.35^{**}, 1.36^*, 1.37^*.

Harder problems

1. I showed in the lecture that the symmetric group S_6 has an outer automorphism, and stated without proof that this group has no outer automorphism for $n \neq 6$. Prove this.

Note that, for n finite, it suffices to prove that, if $n \neq 6$, then any subgroup of index n in S_n is the stabiliser of some point. However, the result is true for infinite as well as finite n, and what must be proved in the infinite case is a bit more complicated.

2. Find an example of a transitive permutation group containing no derangement of prime order.

Open problems

I think that these three open problems are very difficult. Easier open problems will come up later in the course.

1. Find, if possible, an elementary proof (that is, not using the Classification of Finite Simple Groups) of the *Fein–Kantor–Schacher theorem*:

> If n is a transitive permutation group of degree $n > 1$, then G contains a derangement (an element with no fixed points) with prime power order.

2. *Isbell's Conjecture* from 1960 is the following:

> There is a function $f(p,b)$ such that, if $n = p^a \cdot b$ with p a prime not dividing b and $a > f(p,b)$, then any transitive permutation group of degree n contains a derangement of p-power order.

Prove this.

3. Find "improvements" to the *Cameron–Cohen theorem*:

> If G is a transitive permutation group of degree $n > 1$, then the proportion of derangements in G is at least $1/n$.

Of course the bound is best possible; but if G is not sharply 2-transitive then improvements are possible. Some results on this are known.