Chamber systems and buildings

1 Incidence geometry

Incidence geometry, in its most general sense, involves a number of different types of geometric objects, with a binary relation of ‘incidence’ which may hold between objects of different types. The objects may be points, lines, conics, etc.; the usual term for them is ‘varieties’ (borrowed from algebraic geometry). The most common situation is where there are just two types, though the more general case was considered by Moore in 1896.

Formally, an incidence geometry consists of a set V of *varieties*, a finite set I of *types*, a *type map* $\tau : V \to I$, and a symmetric incidence relation $*$ on V, satisfying the following axiom:

(IG1) For $v, v' \in V$, we have $\tau(v) = \tau(v')$ and $v * v'$ if and only if $v = v'$.

In other words, a variety is incident with itself (this is just a convenient convention) and with no other variety of the same type. We denote by V_i the set of varieties of type i, for $i \in I$. The incidence relation $*$ can be regarded as adjacency in a graph, which is multipartite with parts V_i (for $i \in I$), together with a loop at each vertex. The *rank* of a geometry is the number of types.

We have assumed that the rank is finite. This assumption can be relaxed; but, as we will see, induction on the rank is one of the most powerful weapons in a geometer’s arsenal.

A geometry of rank 2 is what is often referred to as an *incidence structure*. Combinatorialists often refer to the two types of varieties in an incidence structure as *points* and *blocks*, and (where possible) like to identify a block with the set of points incident to it. However, from our point of view, a rank 2 geometry is a bipartite graph. This graph is often called the *Levi graph* of the incidence structure, after Levi in 1929.

Sometimes it is possible to change our point of view of an incidence geometry to be closer to that just described in the rank 2 case. Let 0 be a distinguished type. We take the elements of the set V_0 as *points*. Now the *shadow* $\mathrm{Sh}(v)$ of a variety v is the set of all points incident with v. If the geometry has the additional property that distinct varieties have distinct shadows, then we can identify all varieties with sets of points. However, description of incidence in terms of the intersections of shadows is not straightforward, except in special cases. (In projective spaces, which we discuss below, two varieties are incident if and only if the shadow of one contains the shadow of the other; that is, incidence is ‘symmetrised inclusion’.)

Further axioms are generally assumed; these are now fairly standard although this has not always been true. These axioms concern maximal flags and connectedness. As explained in the Introduction, we do not assume these axioms without saying so explicitly.

A *flag* is a set of mutually incident varieties. Note that the type map restricted to a flag is one-to-one, according to our axioms. The *type* $\tau(F)$ of a flag F is the set of types of its varieties, that is, the image of F under the type map. Its *cotype* is $I \setminus \tau(F)$. The *rank* of a flag is its cardinality (or, as is the same, the cardinality of its type), and its *corank* is the cardinality of its cotype.

We make the following assumptions.

(IG2) A maximal flag contains one variety of each type; that is, the type map restricted to a maximal flag is a bijection.

(IG3) A flag of corank 1 is contained in at least two maximal flags.

Condition (IG2) is called the *transversality condition*, since it asserts that any maximal flag is a transversal to the partition of V induced by the type function.

Sometimes condition (IG3) is relaxed, in which case we call a geometry *firm* if it holds. Moreover, a geometry is called *thin* if (IG3) holds with ‘exactly two’ in place of ‘at least two’, and *thick* if (IG3) holds with ‘at least three’ in place of ‘at least two’.

Example A cycle consisting of $2n$ vertices and $2n$ edges is bipartite, and so is a thin rank 2 incidence structure. Somewhat confusingly, it is an *n-gon* (that is, its varieties are the vertices and edges of an ordinary n-gon, and incidence is the usual geometric notion). Figure 1 shows a 4-gon in its usual representation and as a geometry (a Levi graph). In the second diagram, the two types are shown as dots of different sizes. Loops have been omitted.

Example A polyhedron in Euclidean 3-space defines an incidence structure with three types of varieties: vertices, edges and faces.

Let F be a non-maximal flag. The *residue* of F, denoted by $R(F)$, is the subgeometry consisting of all the varieties v satisfying $v * w$ for all $w \in F$ but $v \notin F$. It is a geometry whose type is the cotype of F (and whose rank is the corank of F). If (IG2) and/or (IG3) holds in the whole geometry, then it holds in the residue of each flag.

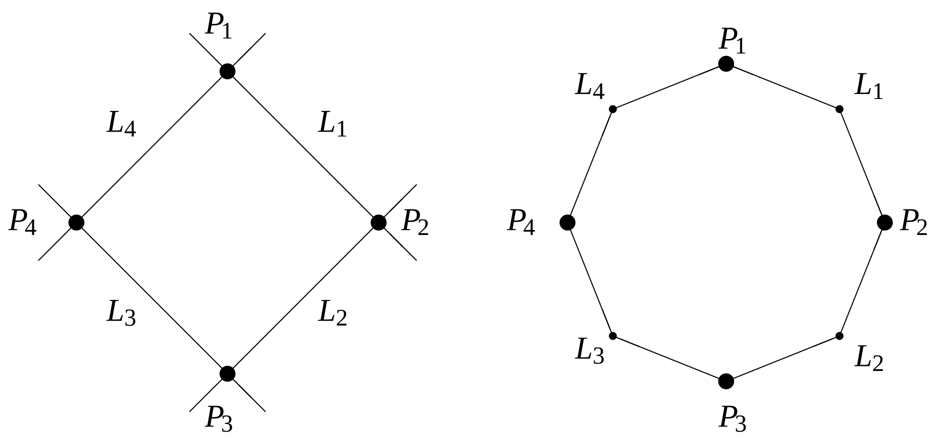

Figure 1: A 4-gon

This construction allows the possibility of induction on the rank of the geometry, both for proofs and for definitions.

A geometry $\mathcal{G}$ is *connected* if the graph on V defined by the incidence relation is connected. We say that $\mathcal{G}$ is *residually connected* if, for any flag F with corank at least 2, the residue of F is connected. (Of course, if (IG3) holds, then the residue of a flag of corank 1 is never connected.) For rank 2 geometries, connectedness and residual connectedness are the same condition. Now the connectedness axiom which is usually assumed is the following.

(IG4) The geometry is residually connected.

We now turn to the construction of geometries from groups. An *automorphism* of a geometry is a permutation of the set V of varieties which preserves both the type function and the incidence relation; that is, every variety is mapped to one of the same type, and incident pairs of varieties are mapped to incident pairs. This is sometimes called a *strong automorphism*, in contrast with a *weak automorphism*, which is also allowed to permute the types. (Thus, a weak automorphism can be described as a pair (g,γ), where g is a permutation of V and γ a permutation of I, satisfying $\tau(vg) = \tau(v)\gamma$ for all $v \in V$ and also $(v * v') \Rightarrow (vg * v'g)$. Thus a weak automorphism is strong if and only if γ is the identity permutation.) We consider only strong automorphisms, and refer to them just as automorphisms.

Any automorphism carries a maximal flag to another maximal flag. A group G of automorphisms of the geometry is said to act *flag-transitively* if any maximal flag can be mapped to any other by some automorphism in G.

The case where the geometry most closely reflects the structure of the group occurs when the group is flag-transitive. In this case, we can describe the geometry within the group, as follows.

Let G be a flag-transitive group of automorphisms of a geometry $\mathcal{G}$. Let F be a maximal flag of $\mathcal{G}$, and let v_i be the unique variety of type i in F and H_i the stabiliser of v_i in G, for all $i \in I$. Now, since G acts transitively on V_i, we can identify the varieties of type i with the right cosets of H_i in G: the variety $v_i g$ corresponds to the coset $H_i g$.

We claim:

> Two varieties are incident if and only if the corresponding cosets have non-empty intersection.

For suppose first that $H_i g_i \cap H_j g_j \neq \emptyset$. If g belongs to this intersection, then $H_i g_i = H_i g$ and $H_j g_j = H_j g$. Now the varieties $H_i g$ and $H_j g$ are the images of H_i and H_j under the automorphism g; since $H_i = v_i$ and $H_j = v_j$ are incident, so are $H_i g$ and $H_j g$.

Conversely, suppose that $H_i g_i$ and $H_j g_j$ are incident. By flag-transitivity, there is an automorphism g carrying H_i and H_j to $H_i g_i$ and $H_j g_j$; so $g \in H_i g_i \cap H_j g_j$.

Now we can reverse this procedure: given a group G and subgroups H_i for $i \in I$, we can define a geometry whose varieties are the cosets of these subgroups, with the obvious type map and with two varieties incident if they have non-empty intersection. We call this a *coset geometry* $\mathcal{G}(G;(H_i : i \in I))$. Clearly (IG1) and (IG2) hold, and G acts by right multiplication as a group of automorphisms of the geometry. We investigate what group-theoretic conditions guarantee the other axioms.

Let $B = \bigcap_{i \in I} H_i$, and for $i \in I$, let $P_i = \bigcap_{j \in I \setminus \{i\}} H_j$. Then B is the stabiliser of our standard maximal flag F, and (by flag-transitivity) the stabiliser of any other maximal flag is a conjugate of B. Moreover, P_i is the stabiliser of the sub-flag of F of cotype $\{i\}$. Then the varieties of type i incident with $F \setminus \{v_i\}$ are the cosets of H_i contained in $H_i P_i$; so

> (IG3) holds if and only if P_i is not contained in H_i.

To investigate residual connectedness, we first note that the coset geometry $\mathcal{G}(G;(H_i : i \in I))$ is connected if and only if the subgroups H_i (for $i \in I$) generate G. Now the residue of H_i in the coset geometry $\mathcal{G}(G,(H_i : i \in I))$ is the coset geometry $\mathcal{G}(H_i,(H_i \cap H_j : j \in I \setminus \{i\}))$. For the residue of H_i consists of all cosets $H_j h$ with $h \in H_i$ and $j \neq i$. Now $H_j h = H_j h'$ if and only if $h'h^{-1} \in H_i \cap H_j$; so the varieties of type j correspond to cosets of $H_i \cap H_j$ in H_i. Two varieties of this residue are incident if and only if they have the form $H_j h$ and $H_k h$ for some $h \in H_i$.

So finally:

Theorem 1 *The coset geometry $\mathcal{G}(G,(H_i : i \in I))$ is residually connected if and only if, for every subset J of I with $|I \setminus J| \geq 2$, we have*

$$\bigcap_{i \in J} H_i = \left\langle \bigcap_{i \in J \cup \{j\}} H_i : j \in I \setminus J \right\rangle.$$

This is not an easy condition to check!

2 Diagrams

The recent revival of interest in incidence geometry with several types has grown from the work of Buekenhout [2] on diagram geometries. Buekenhout gave a simple pictorial method of describing natural axiomatic classes of geometries as follows.

Let I be a finite set. A *diagram* Δ over I consists of a set Δ_{ij} of geometries for each $i, j \in I$ with $i \neq j$. Each geometry in Δ_{ij} has two types of vertices, which we will call ‘points’ and ‘blocks’. It is customary to assume that the geometries in Δ_{ji} are the duals of those in Δ_{ij}, in the sense that they are the same geometries but the attachment of the labels ‘point’ and ‘block’ to the types is reversed.

Diagrams can be represented pictorially. Each class Δ_{ij} is represented by a label on the edge joining i to j in the complete graph on the set I. We take the symbol describing Δ_{ji} to be the typographic reverse of that describing Δ_{ij}.

Now let $\mathcal{G}$ be a geometry with type set I. We say that $\mathcal{G}$ *belongs to the diagram* Δ if, for any flag F of cotype $\{i, j\}$, the residue of F is isomorphic to a geometry in Δ_{ij}, where the isomorphism carries ‘points’ and ‘blocks’ to varieties of types i and j respectively. (This explains why we assume the duality condition above.)

Thus, any diagram gives an axiomatic definition of a class of geometries.

As an example, we discuss projective geometry in some detail. First we define the classes of geometries in our diagrams. First, a *digon* is an incidence structure in which every point is incident with every block. We represent the class of digons by the *absence* of an edge in the diagram. Next, a *projective plane* is an incidence structure in which every two points are incident with a unique block, and any two blocks with a unique point, but no point is incident with every block and no block is incident with every point. This is represented by a single edge without a label. (Note that both these classes are self-dual, and the symbols used for them are the same when reversed.) Recall that a digon or projective plane is *thick* if every point is incident with at least three blocks and dually.

The *projective space* of dimension n over a division ring D is the geometry whose varieties are all the vector subspaces of the vector space D^{n-1} except for $\{0\}$ and D^{n-1}. The type of a subspace is one less than its dimension (as a vector space), and two subspaces V, V' are incident if $V \subseteq V'$ or $V' \subseteq V$. The type set is $I = \{0, 1, 2, \ldots, n-1\}$. We claim that an n-dimensional projective space is represented by the diagram in Figure 2.

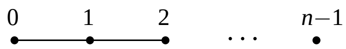

Figure 2: The diagram A_n

To verify this claim, we must calculate the rank 2 residues. Take a flag F containing a subspace V_k of each type $k \in I$ except i and j, where we may suppose that $i < j$. There are two cases:

- $j > i+1$. In this case, the varieties of type i are all the subspaces U of type i satisfying $V_{i-1} \subseteq U \subseteq V_{i+1}$, and those of type j are all the subspaces W of type j satisfying $V_{j-1} \subseteq W \subseteq V_{j+1}$. For any such U and W, we have

  $$U \subseteq V_{i+1} \subseteq V_{j-1} \subseteq W,$$

  so U and W are incident; the residue is a digon.

- $j = i+1$. Then the varieties of types i and $i+1$ are all those subspaces of these types satisfying

  $$V_{i-1} \subseteq U, W \subseteq V_{i+2};$$

  these correspond to the 1- and 2-dimensional subspaces of the 3-dimensional quotient space V_{i+2}/V_{i-1}. Elementary linear algebra shows that this residue is a projective plane.

In fact, the converse holds too. A thick geometry belonging to the diagram A_n above for $n \geq 3$ is a projective space over a division ring. This follows from Hilbert's coordinatisation theorem. This example illustrates how compact the axioms for projective geometry become in this framework.

The convention that digons are represented by the absence of an edge in the diagram helps us to read off properties of the geometry from its diagram. Here are a couple of simple examples.

Theorem 2 *(a) If the diagram of a geometry is disconnected, and two varieties have types in different components, then they are incident.*

(b) Suppose that $0, i, j$ *are types such that the removal of* i *from the diagram leaves* 0 *and* j *in different components. Take varieties of type* 0 *as points. Let* v *and* w *be incident varieties of types* i *and* j *respectively. Then* $\mathrm{Sh}(v) \subseteq \mathrm{Sh}(w)$.

For further details on diagram geometry, see Pasini [4].

3 Chamber systems

Let $\mathcal{P}(\Omega)$ denote the set of all equivalence relations on the set Ω (or, what amounts to the same thing, the set of all partitions of Ω). There is a natural partial order on Ω, which can be defined most simply as the relation of inclusion on the equivalence relations. (We regard an equivalence relation as a set of ordered pairs.) In terms of partitions, the order is given by the rule that $P_1 \leq P_2$ if P_1 *refines* P_2, in the sense that every part of P_1 is contained in a part of P_2. This partial order is a lattice order: the meet of two equivalence relations is just their intersection. The join is more difficult to define. If Π is a set of equivalence relations, the Π-*graph* is the graph with vertex set Ω, in which two vertices are adjacent if and only if they are equivalent with respect to some relation $\pi \in \Pi$. Then the join of π and ρ is the relation whose equivalence classes are the connected components of the $\{\pi, \rho\}$-graph. This lattice is the *lattice of partitions* of Ω. (To simplify notation later, if a set of equivalence relations is indexed, we speak of the I-graph rather than the $\{\rho_i : i \in I\}$-graph.)

Now a *chamber system* of type I on Ω is simply a family $(\rho_i : i \in I)$ of equivalence relations on Ω. The elements of Ω are called *chambers*. We say that chambers α and β are *i-equivalent* if $(\alpha, \beta) \in \rho_i$; sometimes we write this as $\alpha \sim_i \beta$.

We normally impose two conditions which lose no generality. First, we assume that, if $i, j \in I$ with $i \neq j$, and α and β are chambers which are both i-equivalent and j-equivalent, then $\alpha = \beta$. Second, we assume that the I-graph is connected. If this is not so, then we can treat each connected component separately. In terms of the partition lattice, we are assuming that the meet of any two of our relations is equality while the join of all of them is the 'universal' relation $\Omega \times \Omega$.

Now let J be a subset of I. We define a *residue* of type J to be a connected component of the J-graph.

The link with incidence geometries works as follows. Let $\mathcal{G}$ be a geometry satisfying (IG1) and (IG2), with type set I. Take Ω to be the set of maximal flags of $\mathcal{G}$. For each $i \in I$ we define an equivalence relation ρ_i which holds between maximal flags F and F' if and only if the varieties of type j in F and F' are the same for all $j \neq i$. (There is no great loss in assuming (IG2) here since a maximal flag which is not transversal will be invisible from the chamber system viewpoint.)

It is clear that the intersection of two of these equivalence relations is the relation of equality, but their supremum is not determined. We call the geometry *chamber-connected* if the chamber system is connected. How is this notion related to other kinds of connectedness?

Theorem 3 *Let $\mathcal{G}$ be a geometry satisfying (IG1) and (IG2).*

(a) If $\mathcal{G}$ is residually connected, then it is chamber-connected.

(b) If $\mathcal{G}$ is chamber-connected, then it is connected.

(c) Neither of these implications reverses.

Note that, for a rank 2 geometry, the three types of connectedness in Proposition 3 coincide. The geometry is a bipartite graph (the Levi graph of the incidence structure), whose edges are the maximal flags; so its line graph is the chamber graph of the geometry.

Let $\mathcal{G}$ be a geometry with type set I, and let $\mathcal{C}$ be the corresponding chamber system. For any subset J of I, we have two notions of residue in $\mathcal{C}$: a connected component of the J-graph, and the set of chambers in the residue of a flag F of cotype G (that is, the set of maximal flags extending F). We call these *chamber-residues* and *geometric residues* respectively. Since edges in the J-graph join maximal flags agreeing outside J, a chamber-residue is contained in a geometric residue. Proposition 3(a) shows that, if $\mathcal{G}$ is residually connected, the two notions coincide.

This shows that, if a chamber system comes from a residually connected geometry, then we can recover the geometry as follows: the varieties of type i are the chamber-residues of cotype i, and two varieties are incident if they have non-empty intersection. This construction gives the 'most highly connected' geometry for a given chamber system. In particular, it is sometimes possible to start with a geometry which is not residually connected and produce one which is.

Not every chamber system comes from a geometry. A familiar class of examples consists of Latin squares. A *Latin square* of order n may be defined as an $n\times n$ array with entries from $\{1,\ldots,n\}$ with the property that each symbol occurs exactly once in each row or column. Now let Ω be the set of n^2 cells of the array, and define three equivalence relations as follows:

- $(\alpha,\beta)\in\rho$ if α and β lie in the same row;
- $(\alpha,\beta)\in\gamma$ if α and β lie in the same column;
- $(\alpha,\beta)\in\sigma$ if α and β contain the same symbol.

Now each rank 2 residue contains all the cells, and has the structure of an $n\times n$ grid. So the attempted construction of a geometry would yield a single variety of each type.

Chamber systems can be constructed from groups. The construction is in some respects simpler than the construction of geometries; it works 'from the bottom up', rather than 'from the top down'. Let G be a group, B a subgroup of G, and $(P_i : i\in I)$ a family of subgroups each containing B, such that $P_i\cap P_j = B$ for $i\neq j$. We take Ω to be the set of right cosets of B, and, for $i\in I$, two cosets satisfy relation ρ_i if they lie in the same coset of P_i. This defines a chamber system, on which G acts by right multiplication as a group of automorphisms (preserving all the equivalence relations). It is straightforward to show that, for any $J\subseteq I$, the residues of type J correspond to right cosets of the subgroup

$$P_J = \langle P_i : i\in J\rangle;$$

more precisely, a residue is the set of cosets of B in a fixed coset of P_J. The chamber stabiliser B is called the *Borel subgroup* of G, and the subgroups P_J are the *parabolic subgroups*.

We can associate diagrams with chamber systems in much the same way as for incidence geometries. Let Δ_{ij} be a class of rank 2 chamber systems for all distinct $i,j\in I$, where the types in Δ_{ij} are 'points' and 'blocks' and these labels in Δ_{ji} are assigned in the other sense. Then a chamber system belongs to the diagram Δ if its residues of type $\{i,j\}$ belong to Δ_{ij}. For example, in the chamber system constructed from a Latin square, any rank 2 residue is a $n\times n$ grid; so the chamber system has a diagram which is a triangle with each edge labelled 'square grid'.

(Note that a square grid is the chamber system of a generalised digon with equal numbers of varieties of each type.)

This is particularly useful in the case of groups. The *amalgam method* (see [6]) studies groups generated by known subgroups P_i for $i \in I$, which intersect pairwise in a fixed subgroup B. The diagram of the chamber system tells us about the subgroups of G generated by the pairs $\{P_i, P_j\}$. From this, the aim is to get information about G. The methods are technical, and we do not discuss them in detail here.

4 Coxeter groups and buildings

A particularly important class of examples arises from *Coxeter groups*. A Coxeter group is a group defined by a presentation of the form

$$G = \langle x_i \ (i \in I) : x_i^2 = 1 \ (i \in I), \ (x_i x_j)^{m_{ij}} = 1 \ (i, j \in I, i \neq j) \rangle, \qquad (1)$$

where the m_{ij} are integers (at least 2) or ∞. (By convention, if $m_{ij} = \infty$, this relation is absent.) Much is known about Coxeter groups; some of this is summarised below.

Theorem 4 *Let G be the Coxeter group with presentation given by Equation (1).*

(a) The orders of x_i and $x_i x_j$ are 2 *and m_{ij} respectively – that is, not strictly smaller. (If $m_{ij} = \infty$, then $x_i x_j$ has infinite order.)*

(b) For $J \subseteq I$, the subgroup G_J of G generated by $\{x_i : i \in J\}$ is the Coxeter group defined by the presentation

$$G_J = \langle x_i \ (i \in J) : x_i^2 = 1 \ (i \in J), \ (x_i x_j)^{m_{ij}} = 1 \ (i, j \in J, i \neq j) \rangle.$$

(c) G is isomorphic to the group generated by reflections in a family $(H_i : i \in I)$ of hyperplanes in Euclidean or hyperbolic space where the angle between H_i and H_j is π/m_{ij}. (If $m_{ij} = \infty$, the hyperplanes are parallel.)

(d) G is finite if and only if the space in (c) is Euclidean and any two hyperplanes intersect, or equivalently, the matrix $A = (a_{ij})$ with $a_{ii} = 1$ and $a_{ij} = -\cos(\pi/m_{ij})$ for $i \neq j$ is positive definite.

Any Coxeter group is described by a *Coxeter diagram*, having one node for each element of I, and an edge labelled m_{ij} from i to j. By convention, if $m_{ij} \leq 4$ we use instead $m_{ij} - 2$ unlabelled edges from i to j; that is, if $m_{ij} = 2$, we omit the edge, if $m_{ij} = 3$ we put a single edge, and if $m_{ij} = 4$ we put a double edge. Now G is finite if and only if the Coxeter diagram is a disjoint union of diagrams of the types A_n, C_n, D_n, E_6, E_7, E_8, F_4, $I_2^{(m)}$ ($m \geq 5$), H_3 and H_4. The A_n diagram is the same as the one in Figure 2. See Humphreys [3] for more details.

For any Coxeter group, there is an associated *Coxeter complex*, a cell complex constructed as follows. Take the images under G of the reflecting hyperplanes in part (c) of the theorem. These decompose the real vector space into pieces, which are the cells of the complex.

There is also a chamber system, obtained by taking $B = 1$ and $P_i = \langle x_i \rangle$ for $i \in I$. This chamber system is geometrically realised by the Coxeter complex: the chambers are the cells of the complex of maximum dimension, and two cells satisfy one of the relations ρ_i if and only if they are separated by one of the reflecting hyperplanes. In fact, the hyperplanes mentioned in (c) of the theorem bound the *fundamental chamber* C, and the reflection of C in H_i has the relation ρ_i to C. Now G acts regularly on the set of chambers, and so these relations can be transported around the complex by elements of G. We give, as an example, the Coxeter complex for the Coxeter group with presentation

$$\langle x_1, x_2 : x_1^2 = x_2^2 = (x_1 x_2)^4 = 1 \rangle$$

(this is the dihedral group of order 8): see Figure 3.

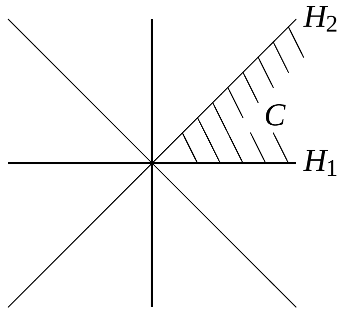

Figure 3: A Coxeter complex

In the figure, the eight chambers are the wedge-shaped regions. Two chambers are in the relation ρ_1 if they are separated by a thick line, and in the relation ρ_2

if they are separated by a thin line. The chamber system comes from a geometry, namely the 4-gon. See Figure 4, which adds the chamber system to Figure 1. The first two diagrams are the same as in the earlier figure: the 4-gon drawn conventionally and as a Levi graph. The third diagram shows the chamber system: its chambers are the edges of the bipartite graph, and two chambers are related by the first or second relation (represented by a thick or thin edge respectively) if they meet in a vertex of the first or second type (a point or a line respectively).

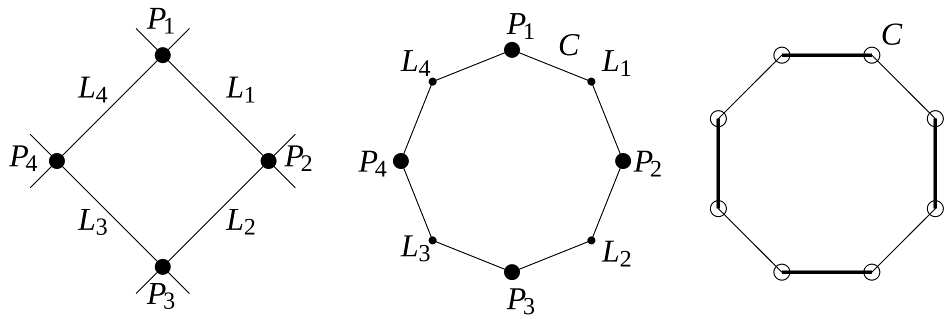

Figure 4: A 4-gon

The most important chamber systems are *buildings*. Indeed, the notion of chamber system was developed by Tits to provide a setting for buildings, which he had previously regarded as incidence geometries. (It was this earlier work of Tits which inspired Buekenhout's definition of diagram geometries.)

Let G be a Coxeter group, with presentation given by Equation 1. A *G-building* is a set C with a function $d : C \times C \to G$ satisfying certain technical conditions that will not be given here. Essentially, d is a 'G-valued metric', and we think of two elements c, c' of C as being 'nearest' when $d(c,c') = x_i$ for some i. The axioms imply that the relation

$$\rho_i = \{(c,c') : d(c,c') \in \{1, x_i\}\}$$

is an equivalence relation; so C has the structure of a chamber system with type set I. The axioms also imply that it has many subsystems (called *apartments*) which are isomorphic to the Coxeter complex of G: in fact, any two chambers lie in an apartment.

For example, a triangle or 3-gon is a Coxeter complex for the Coxeter group

$$\langle x_1, x_2 : x_1^2 = x_2^2 = (x_1x_2)^3 = 1\rangle,$$

Condition	group element
$P' = P, L' = L$	1
$P' = P, L' \neq L$	x_1
$P' \neq P, L' = L$	x_2
$P' \neq P, L' \neq L, P' * L$	x_1x_2
$P' \neq P, L' \neq L, P * L'$	x_2x_1
opposite	$x_1x_2x_1 = x_2x_1x_2$

Table 1: A 3-gon

the dihedral group of order 6. If the flag $\{P,L\}$ is indexed by the identity, then the indexing of the six flags $\{P',L'\}$ in the triangle is shown in Table 1.

We call two flags in a triangle *opposite* if no equalities or incidences hold between any of their members. (The meaning of the term is clear from a picture of the chamber system.) Now, in a projective plane, we can use the table to assign the values of the G-valued metric d to pairs of flags; the result is a building, whose apartments are its triangles. It is an easy exercise to show that a rank 2 incidence geometry is a projective plane if and only if

(a) given any flag, there is a flag opposite to it;

(b) two opposite flags are contained in a unique triangle.

(Condition (b) shows that two distinct points are incident with a unique line and dually.)

This observation can be extended. The Coxeter group of type A_n (see Figure 2) is isomorphic to the symmetric group of degree $n+1$. Now the buildings associated with this Coxeter group are precisely the n-dimensional projective spaces.

The finite buildings of rank at least 3 (and, more generally, the buildings of rank at least 3 whose Coxeter groups are finite – these are the so-called *spherical buildings*) have been classified by Tits [7]. All of these buildings arise from geometries, much as for type A_n in the preceding paragraph.

For further details on buildings, see Brown [1] or Ronan [5].

References

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