# Donald at Queen Mary: Climbing walls and PLRs

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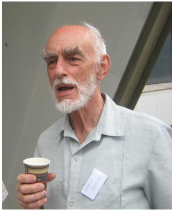
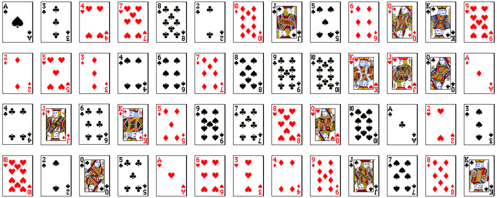
Donald Preece Memorial Day, 17 September 2015





# Our first meeting

I first met Donald at the BCC in Aberystwyth in 1973, which I think was his first BCC, perhaps his first combinatorics conference. Rosemary has told this story. After his talk, I sat next to him on the excursion coach, and the result of that discussion was a joint publication constructing some designs resembling the one on the title page of these slides.

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This is a table from the paper (mentioned by Rosemary in her talk). I didn't understand the exact relation between Donald's and my points of view for more than 20 years, when I found an infinite family of these designs.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| AAA | BBB | CCC | DDD | EEE | FFF | GGG | HHH | III | JJJ | KKK | LLL | MMM | NNN | OOO | PPP |
| HGB | GHA | FED | EFC | DCF | CDE | BAH | ABG | POJ | OPI | NML | MNK | LKN | KLM | JIP | IJO |
| KJD | LIC | ILB | JKA | ONH | PMG | MPF | NOE | CBL | DAK | ADJ | BCI | GFP | HEO | EHN | FGM |
| MIE | NJF | OKG | PLH | IMA | JNB | KOC | LPD | EAM | FBN | GCO | HDP | AEI | BFJ | CGK | DHL |
| OFL | PEK | MHJ | NGI | KBP | LAO | IDN | JCM | GND | HMC | EPB | FOA | CJH | DIG | ALF | BKE |
| PCN | ODM | NAP | MBO | LGJ | KHI | JEL | IFK | HKF | GLE | FIH | EJG | DOB | CPA | BMD | ANC |
| BDC | ACD | DBA | CAB | FHG | EGH | HFE | GEF | JLK | IKL | LJI | KIJ | NPO | MOP | PNM | OMN |
| CKI | DLJ | AIK | BJL | GOM | HPN | EMO | FNP | KCA | LDB | IAC | JBD | OGE | PHF | MEG | NFH |
| DMP | CNO | BON | APM | HIL | GJK | FKJ | ELI | LEH | KFG | JGF | IHE | PAD | OBC | NCB | MDA |
| EBF | FAE | GDH | HCG | AFB | BEA | CHD | DGC | MJN | NIM | OLP | PKO | INJ | JMI | KPL | LOK |
| FLO | EKP | HJM | GIN | BPK | AOL | DNI | CMJ | NDG | MCH | PBE | OAF | JHC | IGD | LFA | KEB |
| GPJ | HOI | ENL | FMK | CLN | DKM | AJP | BIO | OHB | PGA | MFD | NEC | KDF | LCE | IBH | JAG |
| IEM | JFN | KGO | LHP | MAI | NBJ | OCK | PDL | AME | BNF | COG | DPH | EIA | FJB | GKC | HLD |
| JOH | IPG | LMF | KNE | NKD | MLC | PIB | OJA | BGP | AHO | DEN | CFM | FCL | EDK | HAJ | GBI |
| LNG | KMH | JPE | IOF | PJC | OID | NLA | MKB | DFO | CEP | BHM | AGN | HBK | GAL | FDI | ECJ |
| NHK | MGL | PFI | OEJ | JDO | ICP | LBM | KAN | FPC | EOD | HNA | GMB | BLG | AKH | DJE | CIF |

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But his biggest contribution occurred in 1999. The committee found itself without a conference venue, due to circumstances beyond our control. Donald stepped in and, with John Lamb's help, organised a very successful BCC at the University of Kent at Canterbury.

# Donald at Queen Mary

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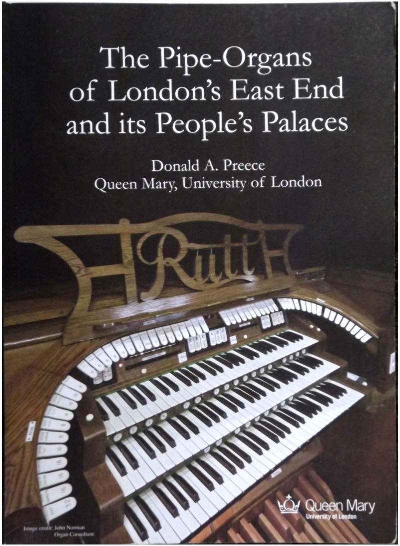
He also became involved with the Luncheon Club at Queen Mary, and through this, became involved with the Organ in the Great Hall, which was then in very poor condition. He was very much concerned with the refurbishment of the organ, and one of his compositions was played at its re-inauguration in 2013.

This is the cover of Donald's remarkable survey of East End organs, published by QMUL in 2012.
The cover picture shows the console of the refurbished organ, which we will hear later this afternoon.
His two copies of the book are both heavily annotated …



# Terraces, daisy chains, tredoku and more

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At my retirement conference in 2013, he posed various challenges concerned with tredoku, a 3-dimensional version of Sudoku which appeared in *The Times*. Afterwards, quite a few members of the audience could be seen trying their hand at these.

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A primitive root modulo an integer $n$ is an integer $r$ which is coprime to $n$ and has the property that every integer coprime to $n$ is congruent to a power of $r$. For example, 3 is a primitive root mod 5, since $3^1 \equiv 3$, $3^2 \equiv 4$, $3^3 \equiv 2$, and $3^4 \equiv 1$.

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Primitive roots do not exist for every integer: only numbers which are an odd prime power, twice an odd prime power, or 4 have them.

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I will give one of Donald’s constructions which shows how he ingeniously bridged the gap. The next slide is in Donald’s words.

Consider the following sequence of the elements of $\mathbb{Z}_{35}$:

START

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0

25 20 30 32 26 8 24 2 6 18 19 22 31 23 34 14 28 ↙

FINISH

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The last 17 entries, in reverse order, are the negatives of the first 17, which, with the zero, can also be written

$$5^5 \quad 5^6 \quad 5^7 \mid 3^1 \quad 3^2 \quad 3^3 \quad 3^4 \quad 3^5 \quad 3^6 \quad 3^7 \quad 3^8 \quad 3^9 \quad 3^{10} \quad 3^{11} \quad 3^{12} \mid 7^4 \quad 7^5 \mid 0.$$

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If we write the respective entries here as $x_i$ $(i = 1, 2, \ldots, 18)$, then the successive differences $x_{i+1} - x_i$ $(i = 1, 2, \ldots, 17)$ are

$$5 \quad -10 \quad -2 \quad 6 \quad -17 \quad -16 \quad -13 \quad -4 \quad -12 \quad -1 \quad -3 \quad -9 \quad 8 \quad -11 \quad -15 \quad -14 \quad -7.$$

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| 10 | 15 | 5 | 3 | 9 | 27 | 11 | 33 | 29 | 17 | 16 | 13 | 4 | 12 | 1 | 21 | 7 | $\searrow$ |
| | | | | | | | | | | | | | | | | | 0 |
| 25 | 20 | 30 | 32 | 26 | 8 | 24 | 2 | 6 | 18 | 19 | 22 | 31 | 23 | 34 | 14 | 28 | $\swarrow$ |
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Ignoring minus signs, these differences consist of each of the values $1, 2, \ldots, 17$ exactly once. This is a special type of terrace.

Carmichael's lambda-function $\lambda(n)$ is the maximum order of an element in the group of units of $\mathbb{Z}_n$, the integers mod $n$. (That is, the largest number of distinct powers we can get modulo $n$ from a fixed element coprime to $n$.) An element of the group of units $U_n$ is a primitive lambda-root if its order is $\lambda(n)$.

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In the preceding example, $\lambda(35)$ is the least common multiple of $\lambda(5) = 4$ and $\lambda(7) = 6$, that is, $\lambda(35) = 12$. Now 3 is a primitive lambda-root mod 35: its powers mod 35 are

$$\begin{array}{llllll} 3^1 = 3, & 3^2 = 9, & 3^3 = 27, & 3^4 = 11, & 3^5 = 33, & 3^6 = 29, \\ 3^7 = 17, & 3^8 = 16, & 3^9 = 13, & 3^{10} = 4, & 3^{11} = 12, & 3^{12} = 1. \end{array}$$

Motivated by this, Donald and I embarked on a study of primitive lambda-roots. We never found a suitable place to publish it, but you can access the notes (and the GAP functions I wrote for computing with them) at `https://cameroncounts.wordpress.com/lecture-notes/`

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`https://cameroncounts.wordpress.com/lecture-notes/`
(I should add that I never persuaded Donald to use the computer to do these calculations: he worked on paper on the train journey to London from East Malling, and presented me with his findings and his challenges, when he arrived.)

The notes are mainly expository, and contain many open problems. There are some unexpected connections. For example, if $\lambda^*(m)$ is the greatest $n$ such that $\lambda(n) = m$, then $\lambda^*(2m)$ is also the denominator of the Bernoulli number $B_{2m}$, re-scaled. We give a proof, but I don't really understand why. (In fact, we found the key in a paper on mathematical physics!)

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$$U_n = \langle x \rangle_a \times \langle y \rangle_b \times \langle z \rangle_c$$

to denote that $U_n$ is the direct product of cyclic subgroups generated by $x, y, z$, and that the orders of these elements are $a, b, c$ respectively.

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where the orders as well as the generators themselves are in arithmetic progression; and

$$U_{455} = \langle 92 \rangle_4 \times \langle 93 \rangle_{12} \times \langle 94 \rangle_6,$$

where the generators are consecutive and the orders are even.

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**Theorem**

*Let $n$ be a prime congruent to 7 or 31 (mod 36), $n > 7$. Suppose that the roots $x_1$ and $x_2$ of $x^2 + 3x + 3 = 0$ in $\mathbb{Z}_n$ have orders $(n-1)/2$ and $n-1$ respectively. Then*

$$U_n = \langle 2x_2 + 3 \rangle_m \times \langle x_2 + 1 \rangle_3 \times \langle -1 \rangle_2,$$

*where $m = (n-1)/6$.*

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This and two similar theorems covered all cases of three generators in AP with orders 2, 3 and $(n-1)/6$ when $n$ is prime.

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We remarked that we had been unable to find decompositions with more than four terms; this is an open problem.

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