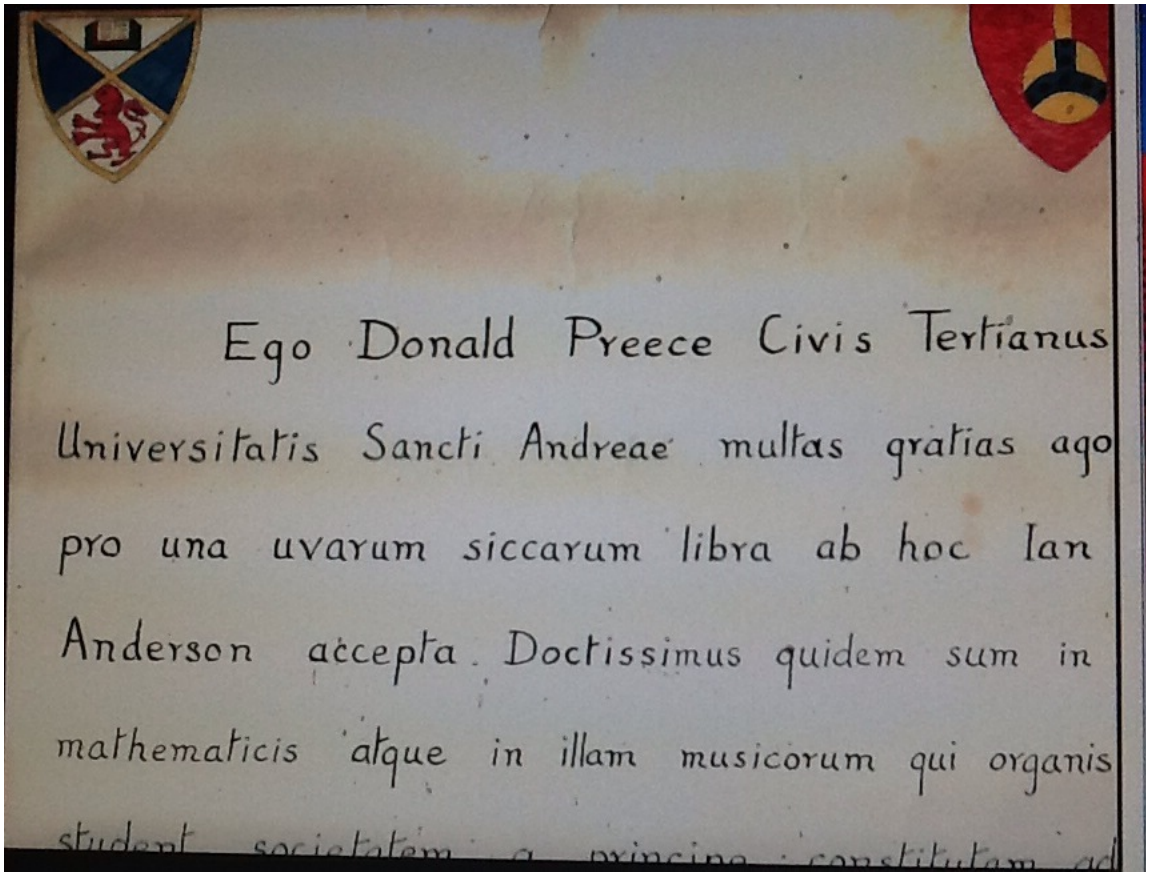
Donald Preece Memorial Day

Ian Anderson, University of Glasgow

I first met Donald Preece 55 years ago when, in the autumn of 1960, I became a first year student at the University of St Andrews. First year students were known as Bejants, and there was a tradition that each Bejant had a Senior Man, a third or fourth year student who would help him settle down in the strange new world of university. Donald had found out that a mathematics student from Edinburgh was to be living in his residence, St Regulus Hall; that student was me, and he offered to be my Senior Man. Donald had been at George Heriot's School in Edinburgh, whereas I had been at the Royal High School, but that didn't matter.

One of the traditions at St Andrews was the celebration of Raisin Monday, which recalled past times when students went home to refurbish their supplies of oatmeal, raisins etc to see them through the rest of the term. By 1960 the custom was that each bejant gave a pound of raisins (or something stronger!) to their senior man who would in return give a receipt in Latin. The bejant had to carry the receipt around with him all day, and produce it if challenged. The penalty for not producing it, or for errors in it, was to have to sing a verse of *Gaudeamus Igitur*, or something worse, like a ducking.

I still have Donald's receipt – see picture below. It describes his interests; mathematics and music in particular. A lot has already been said about his music: he started young, and I have book prize that he received at the age of 12 for singing. Another love he had at school was that of languages, and I later discovered that the person who taught him French at school was none other than my father's closest lifelong friend. Donald described him to me as an inspirational teacher. In later life he kept up his French sufficiently well to actually lecture in French on several occasions.


Ego Donald Preece Civis Tertianus Universitatis Sancti Andreae multas gratias ago pro una uvarum siccarum libra ab hoc Ian Anderson accepta. Doctissimus quidem sum in mathematicis atque in illam musicorum qui organis student societatem a principio constitutam ad

Latin receipt given to me by Donald, Raisin Monday 1960

Donald's niece Heather came to study mathematics at Glasgow, but I lost touch with Donald for many years as he pursued a career in Statistics. However we eventually renewed contact at the British Combinatorial Conferences. I was aware of his interest in combinatorial designs, particularly those related to the design of experiments, and in retrospect wish that I had tried to collaborate with him earlier than I did, as in fact our interests were very close. What actually brought us together as co-authors was a request I received in the year 2000 from Tom Aitchison of the Statistics department in Glasgow to construct a schedule for a series of tests on patients where among other things it was important to have a balance in carry over effects – in other words each treatment should follow each other equally often. The Statistics department had good contacts with Glasgow's Western Infirmary, and this problem had arisen from a particular trial. I solved the problem, wrote it up and sent it to Donald for his comments, in case it was all known already. Donald responded enthusiastically, came through to Glasgow during a stay at his mother's house in Edinburgh, made suggestions and additions, and so our first joint paper was eventually published in Utilitas in 2002. Here is one of the designs…..

1273645	1726354	
2314756	2137465	
3425167	3241576	
4536271	4352617	Group 1
5647312	5463721	
6751423	6574132	
7162534	7615243	
1365472	1634527	
2476513	2745631	
3517624	3156742	
4621735	4267153	Group 2
5732146	5371264	
6143257	6412375	
7254361	7523416	
1457236	1542763	
2561347	2653174	
3672451	3764215	
4713562	4175326	Group 3
5124673	5216437	
6235714	6327541	
7346125	7431652	

Here there are 42 patients each undergoing trials 1,…,7 in some order. The patients are in three groups of 7, such that:

i) In each group, each pair i and j occurs in exactly two rows, once in the first half of a row and once in the second half;
ii) For each i, the 42 pairs in positions i and i+1 are all distinct;
iii) The sets consisting of the first three entries in each row form a balanced incomplete block design;
iv) The sets consisting of the first four entries in each row form also form a BIBD.

This design also has the properties

v) each ordered triple ijk of distinct treatments occurs in just one row;
vi) The six arrays give a complete set of mutually orthogonal squares.

Basic to the construction is the fact that the first rows of the six cyclic arrays are terraces. I'll return to terraces shortly, but mention of Donald visiting me in Glasgow suggests that at this point I should say a little about some of his other visits. Donald's interest in organs has already been discussed by other speakers. I can add that he made two visits to hear Glasgow organs – one, a newly constructed Bach organ which had been presented to the University of Strathclyde who have since then disgracefully ignored it; the other, the 1937 Wurlitzer cinema organ that was originally in the Ritz cinema, Stockport, but is now in Glasgow in Pollokshaws Burgh Hall. But, more personally, I recall that when he discovered that my three children were all musically gifted and aiming towards music as a career, he took great interest in them, and on several occasions came through to our Glasgow home to accompany them on the piano. Donald was a very good sight reader! My eldest son Robert plays the cello, as Donald had done in his youth, and it somehow seemed the most natural thing in the world that the elderly lady who accompanied Robert in his final recital for his Master's degree at the RSAMD was the same lady who had accompanied Donald when he was doing his cello exams as a schoolboy nearly 50 years earlier.

Donald took great interest in their progress, and when my second son Ian William completed his studies at the Royal Academy of Music in London, gaining the viola prize, Donald dragged Peter and Rosemary along from Queen Mary to hear him perform Bartok's viola concerto. Donald gave many of his chamber music scores to me to distribute among my children, music he and his family had used when they played together during his own childhood.

The basis of much music is to present a theme, develop it and produce variations which show the hidden delights of structure and elegance lying within the original idea. I think that '**variations on a theme**' also describes the work I did with Donald on Terraces.

Terraces had been used for some time in the construction of designs, and the terminology in fact dates back to a 1984 paper of Rosemary Bailey. A terrace mod n contains each number mod n once, and the differences between consecutive entries , ignoring + and – signs, give each of 1,…, (n-1)/2 twice if n is odd, and each of 1,…, (n-2)/2 twice and n/2 once if n is even. Here are some examples

Mod 6: 2 5 3 4 6 1
Differences 3 -2 1 2 1

Mod 9: 0 1 8 2 7 3 6 4 5 narcissistic
Differences 1 -2 3 -4 -4 3 -2 1 reflective

Mod 10: 0 1 9 2 8 3 7 4 6 5 directed
Differences 1 -2 3 -4 5 4 -3 2 -1

Mod 11 : 1 2 4 8 5 10 9 7 3 6 0 half and half, da capo
Differences 1 2 4 -3 5 -1 -2 -4 3 5

Note Donald's use of the mathematical term '*Da Capo*' describing the behaviour of the differences, going back to the beginning and starting again.
Narcissistic refers to the *reflective* property of the differences; *directed* indicates that for each i, both i and –i appear in the differences; and *half and half* indicates that each difference number appears once in each half. As other speakers have said, Donald took great care over definitions; they were to help the reader in his or her understanding.

At the BCC in 2001 at the University of Sussex, Donald presented me with a thick folder consisting of page after page of terraces constructed by many different methods. There were some very elegant examples, and in many cases there was clearly a general method which seemed to work for many different lengths of terraces. Donald had a remarkable intuition which produced very fruitful approaches, and it was my job to provide proofs wherever possible, and indeed to find out what the theorem should be (which, in some cases, was not exactly clear). Donald's basic idea in these constructions was to use sequences of powers of a given number. This idea is most simply seen in the example of mod 11 using powers of 2, 2 being a primitive root:

1 2 4 8 5 10 9 7 3 6 0

with differences

1 2 4 -3 5 -1 -2 -4 3 5

Donald's variations on this theme were manifold and elegant. Sometimes his methods seemed to ask for too much. In constructing terraces of length a multiple of 3, he produced a construction which I found depended on a prime number having two primitive roots x and y satisfying the condition

$$(2x - 1)\,(y - 1) = \pm\, y \mod p$$

This might seem wishful thinking, but I knew that a colleague of mine at Glasgow, Steve Cohen, an expert in finite fields, had proved a theorem that said, as a special case, that any sufficiently large prime has two primitive roots u , v satisfying u = 2v-2 mod p. If p=1 (mod 4), take x = -1/u and y = v. Then x and y are primitive roots and

$$\begin{aligned}(2x-1)(y-1) &= (-2/u - 1)(v-1) = -\,(2+u)\,(v-1)\,/\,u \\ &= -(2+u)/2 \,.\, 2(v-1)\,/u = -v \,.\, u\,/\,u = -v = -y.\end{aligned}$$

A similar argument works for p = 3 (mod 4).

Donald had great ideas for variations on the theme. I'd like to mention two of them. His first idea was to use arithmetic mod n to construct terraces of lengths n-1, n+1, n-2, n+ 2.

Start with the powers of 2 mod 11

1 2 4 8 5 10 9 7 3 6.

Working mod 10, they provide a terrace !

We have already seen that we can also get a terrace mod 11 from it, either by

1 2 4 8 5 10 9 7 3 6 0

or

6 1 2 4 8 5 10 9 7 3 0.

But then we can get a terrace mod 12 from it:

6 1 2 4 8 11 5 10 9 7 3 0

and then a terrace mod 13

12 6 1 2 4 8 11 5 10 9 7 3 0.

Also, going back to the beginning and going the other way, removing the 10 and reversing the right half we get

1 2 4 8 5 6 3 7 9

which is a terrace mod 9.

These ideas led to many papers, but they are beyond the scope of this talk!

Throughout, Donald's emphasis was on **elegance**. Here, for example, is a terrace of length 280 constructed using arithmetic mod 281.

```
            2            2           2          2
3  ←--------- | 9 ←------- | 27 ------→ | 1 ----→
```

where the 2 above the arrow indicates that each term is obtained from the previous one by multiplying by 2.

Another elegant idea was that of imitating nested Russian dolls, known as **Matryoshka** dolls, each one inside a larger. Start with a terrace mod 3:

1 0 2

Embed it, multiplied by 3, in a terrace mod 9:

```
2 1 5 3 0 6 4 8 7
      --------
```

Now multiply this by 5 and embed it in a terrace mod 45 by surrounding it with the units mod 45 and the multiples of 9: we get what Donald called a MATRYOSHKA TERRACE mod 45:

```
            2        2
27 9 | 6 ----→3| 2 ---→1 | 10 5 25 15 | 0 | negatives mod 45
                          ------------------------------
```

Matryoshka dolls

A second variation of the terrace theme was perhaps less understandable and more mysterious..

Take the following terrace mod 6, for example

4 3 6 2 1 5 (L)

Take a primitive root of 7, say 5, and, working mod 7, evaluate 5 to the above powers. This gives

2 6 1 4 5 3 (E)

which is also a terrace**!**

We call L a *logarithmic* terrace and E an *exponent* terrace. Remarkably there are infinitely many such examples. Indeed if p is a prime of the form 4k+3, then all you need to do to get an exponent terrace of length 4k+2 is to start with

2k+1 1 2k 2 2k-1 3 ….k k+1

and then follow it with the negatives in reverse order. Thus, for example, taking k=2,

5 1 4 2 3 8 9 7 10 6 (E1)

is a terrace and if we replace each by the power of 2 which gives it – for example replacing 8 by 3 - , we get the logarithmic terrace

4 10 2 1 8 3 6 7 5 9

Just to complete my comments on elegance, I should comment on the significance of these exponent terraces – they can be interpreted as terraces in a multiplicative group also! Take (E1) for example and instead of considering differences consider quotients in the group of nonzero numbers mod 11. The quotient 1/5 is 9 since 5 x 9 = 45 = 1 mod 11. The quotient 2/4 is 6 since 4 x 6 = 24 = 2 mod 11. In this way we get quotients, all different, all the non-identity members of the group

9 4 6 7 10 8 2 3 5

So (E1) is a terrace both in the additive group mod 10 and in the multiplicative group of nonzero numbers mod 11.

I leave the last words of this talk to Donald. In his enthusiasm he could be rather flowery in his language, but I thought that the following paragraph, which involves music, French and mathematics, and which he wrote at the end of our second joint paper (Power-sequence terraces for Zn where n is an odd prime power, *Discrete Mathematics* 261 (2003), 31-58), provides a fitting end to this memorial.

"Outside of mathematics, terraces have featured in the temples of the ancients, in subsistence agriculture, in urban housing (whence Professor R. A. Bailey obtained the name), and in landscape gardening. Our terraces add to these assets. We may not have produced rivals to the splendour of *La terrasse des audiences du clair de lune* (Debussy, Preludes pour Piano, no. 7, 1913) – the terrace where the Moonlight holds court – but we have given a sighting of the great elegance that can be achieved using power sequences to construct terraces for Zn."

Glasgow
17 September 2015