Block intersection polynomials (and their applications to graphs and block designs)

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Block intersection polynomials (invented by Peter J. Cameron and LHS) give useful information on the feasible solutions to integer programming problems of a certain type which arise in the study of graphs and block designs having certain regularity properties.

I shall define block intersection polynomials, and give some examples of the theory of these polynomials and their applications to the studies of edge-regular graphs, amply regular graphs, and t-designs.

All graphs in this talk are finite and undirected, with no loops and no multiple edges.

Some definitions

- A graph Γ is *edge-regular* with *parameters* (v, k, λ) if Γ has exactly v vertices, is regular of degree k, and every pair of adjacent vertices have exactly λ common neighbours.

- A graph is *amply regular* with *parameters* (v, k, λ, μ) if it is edge-regular with parameters (v, k, λ) and every pair of vertices at distance 2 have exactly μ common neighbours.

- A graph is *strongly regular* with *parameters* (v, k, λ, μ) if it is edge-regular with parameters (v, k, λ) and every pair of distinct nonadjacent vertices have exactly μ common neighbours (so in particular, every strongly regular graph is amply regular).

- A *clique* in a graph is a set of pairwise adjacent vertices.

- A *block design* is an ordered pair $(V, \mathcal{B})$, such that V is a finite non-empty set, whose elements are called *points*, and $\mathcal{B}$ is a finite non-empty multiset of subsets of V called *blocks*.

- For t a non-negative integer and v, k, λ positive integers with $t \leq k \leq v$, a t-(v, k, λ) *design* (or simply a t-*design*) is a block design with exactly v points, such that each block has size k and each t-subset of the point-set is contained in exactly λ blocks.

- The *incidence graph* of a block design D is the graph whose vertices are the points and blocks of D (including repeated blocks), with $\{\alpha, \beta\}$ an edge precisely when one of α and β is a point and the other is a block containing that point.

For example, the block design

$$Z := (V, \mathcal{B})$$

with point set

$$V := \{1, \ldots, 8\},$$

and block multiset $\mathcal{B} :=$

[1234, 1238, 1256, 1357, 1458, 1467, 1678, 2367, 2457, 2468, 2578, 3456, 3478, 3568]

is a 2-$(8, 4, 3)$ design.

Now, let Γ be a graph, and let S and Q be given vertex-subsets of Γ, with $s := |S|$.

We are interested in using regularity properties of Γ and information on the subgraph induced on S to obtain information about the number n_i of vertices in Q adjacent to exactly i vertices in S ($i = 0, \ldots, s$), sometimes with the aim of obtaining a contradiction to show that no triple (Γ, S, Q) can exist with the given properties.

For $T \subseteq S$, define λ_T to be the number of vertices in Q adjacent to every vertex in T, and for $0 \le j \le s$, define

$$\lambda_j := 1/\binom{s}{j} \sum_{T \subseteq S, |T|=j} \lambda_T.$$

For example, if Γ is an edge-regular graph with parameters (v, k, λ), S an s-clique of Γ with $s \ge 2$, and $Q := V(\Gamma) \setminus S$, then

$$\lambda_0 = v - s, \quad \lambda_1 = k - s + 1, \quad \lambda_2 = \lambda - s + 2.$$

For another example, if Γ is the incidence graph of a t-(v, k, λ) design D, S the set of vertices of Γ consisting of the points on some block B of D, and Q the set of vertices of Γ corresponding to the blocks of D, then n_i is the number of blocks of D meeting B in exactly i points, and for $j = 0, \ldots, t$, $\lambda_j = \lambda_j(D) = \lambda\binom{v-j}{t-j}/\binom{k-j}{t-j}$, the (constant) number of blocks of D containing a j-subset of the point-set.

For each known λ_j, we have the equation:

$$\sum_{i=0}^{s} \binom{i}{j} n_i = \binom{s}{j} \lambda_j. \tag{1}$$

Theorem (with PJC) For k a non-negative integer, define the polynomial

$$P(x,k) := x(x-1)\cdots(x-k+1),$$

let s and t be integers, with $s \geq t \geq 0$, let $n_0, \ldots, n_s$, $m_0, \ldots, m_s$, and $\lambda_0, \ldots, \lambda_t$ be real numbers, and suppose that for $j = 0, \ldots, t$, equation (1) holds. Then

$$\sum_{i=0}^{s} P(i-x,t)(n_i - m_i) =$$

$$\sum_{j=0}^{t} \binom{t}{j} P(-x, t-j)[P(s,j)\lambda_j - \sum_{i=j}^{s} P(i,j) m_i]. \tag{2}$$

We call (2) the *block intersection polynomial* for the sequences $[m_0, \ldots, m_s]$, $[\lambda_0, \ldots, \lambda_t]$, and denote this polynomial by

$$B(x, [m_0, \ldots, m_s], [\lambda_0, \ldots, \lambda_t]).$$

The preceding theorem can be applied to prove:

Theorem Let Γ be a graph, let S and Q be vertex-subsets of Γ, with $s := |S|$, and let $m_0, \ldots, m_s$ be non-negative integers with either $m_i \leq n_i$ for all i or $m_i \geq n_i$ for all i, where n_i is the number of vertices in Q adjacent to exactly i vertices in S.

Let t be an **even** integer with $0 \leq t \leq s$, and for $j = 0 \ldots, t$, let $\lambda_j := 1/\binom{s}{j} \sum_{T \subseteq S, |T|=j} \lambda_T$, where λ_T is the number of vertices in Q adjacent to every vertex in T.

Now, let $B(x) := B(x, [m_0, \ldots, m_s], [\lambda_0, \ldots, \lambda_t])$. Then:

- $B(x) \equiv 0$ if and only if $m_i = n_i$ for all i; otherwise, $B(x)$ is a degree t polynomial with integer coefficients.

- $B(m) \geq 0$ for every integer m if $m_i \leq n_i$ for all i, and $B(m) \leq 0$ for every integer m if $m_i \geq n_i$ for all i.

- $B(m) = 0$ for some integer m if and only if $m_i = n_i$ for all $i \notin \{m, m+1, \ldots, m+t-1\}$, in which case $[n_0, \ldots, n_s]$ is uniquely determined by $[m_0, \ldots, m_s]$ and $[\lambda_0, \ldots, \lambda_t]$.

Example of bounding clique-size in an edge-regular graph

The strongly regular graphs with parameters $(37, 18, 8, 9)$ include Paley(37), but not all strongly regular graphs with these parameters are known. The complement of such a graph (and such a graph) has least eigenvalue $\tau \approx -3.541$, and so the Hoffman bound gives an upper bound of $6 = \lfloor 37/(1 - 18/\tau) \rfloor$ on the size of a clique.

Now let Γ be any edge-regular graph with parameters $(37, 18, 8)$, and suppose that Γ contains a clique S of size 6. We calculate $B(x) := B(x, [0^7], [31, 13, 4]) = 31x^2 - 125x + 120$, and find that $B(2) = -6$. Hence Γ contains no clique of size 6.

I do not know whether there is some edge-regular graph with parameters $(37, 18, 8)$ and a clique of size 5. The size of a maximum clique in Paley(37) is 4.

Application to amply regular graphs

Theorem Let Γ be an amply regular graph with parameters (v, k, λ, μ), and suppose Δ is an induced subgraph of Γ, where Δ has $s \geq 2$ vertices and vertex-degree sequence $[d_1, \ldots, d_s]$. Further suppose that Δ is connected with diameter at most 2 if Γ is not strongly regular. Let $B(x) := x(x+1)(v-s) - 2xsk + (2x+\lambda-\mu+1)\sum_{i=1}^{s} d_i + s(s-1)\mu - \sum_{i=1}^{s} d_i^2$.

Then $B(m) \geq 0$ for every integer m.

Moreover, $B(m) = 0$ for some integer m if and only if each vertex not in Δ is adjacent to exactly m or $m+1$ vertices of Δ, in which case exactly $B(m+1)/2$ vertices not in Δ are adjacent to just m vertices of Δ.

Example

Let Γ be a strongly regular graph with parameters $(76, 30, 8, 14)$. It is unknown whether such a graph exists, although these are "feasible" parameters for a strongly regular graph.

Now suppose Γ contains an induced subgraph Δ isomorphic to (the 1-skeleton of) an octahedron, i.e. the strongly regular graph with parameters $(6, 4, 2, 4)$. Then Δ has $s = 6$ vertices and vertex-degree sequence $[4^6]$. We calculate $B(x)$ as in the Theorem above, and determine that

$$B(x) = 70(x - 2)(x - 51/35).$$

In particular, $B(2) = 0$. Hence, exactly $B(3)/2 = 54$ vertices not in Δ are adjacent to exactly 2 vertices of Δ, and the remaining 16 vertices not in Δ are adjacent to exactly 3 vertices of Δ.

Example of bounding the multiplicity of a block in a t-design

Suppose D is a 4-$(23, 8, 6)$ design (designs with these parameters exist). Further suppose that D has a block B of multiplicity 3 or more. Then there are at least 3 blocks meeting B in 8 points.

Now let

$$\Lambda := [\lambda_0(D), \ldots, \lambda_4(D)] = [759, 264, 84, 24, 6],$$

and calculate

$$B(x) := B(x, [0^8, 3], \Lambda)$$

$$= 36(21x^4 - 106x^3 + 291x^2 - 366x + 140).$$

Since $B(1) = -720$, we conclude it is impossible for a block of D to have multiplicity 3 or more, and so each block of a 4-$(23, 8, 6)$ design can have multiplicity at most 2.

This also shows that each block of a 5-$(24, 9, 6)$ design (such designs exist) can have multiplicity at most 2.

Example for a resolvable t-design

It is unknown whether there exists a 2-$(55, 11, 5)$ design, but we can show that in such a design, each block has multiplicity at most 2.

Suppose now D is a resolvable 2-$(55, 11, 5)$ design. (A block design is *resolvable* if its blocks can be partitioned into parallel classes, a *parallel class* being a set of blocks partitioning the point set.) Further suppose that D has a block B of multiplicity 2 or more. Then there are at least 2 blocks meeting B in 11 points and at least 8 blocks meeting B in no points.

Now let

$$\Lambda := [\lambda_0(D), \lambda_1(D), \lambda_2(D)] = [135, 27, 5],$$

and calculate

$$B(x) := B(x, [8, 0^{10}, 2], \Lambda) = 5(25x^2 - 85x + 66).$$

Since $B(2) = -20$, we conclude that no block of a resolvable 2-$(55, 11, 5)$ design has multiplicity 2 or more. In other words, each resolvable 2-$(55, 11, 5)$ design is simple.

Finally, here is a new theoretical application of block intersection polynomials to the study of t-designs.

Theorem Let t be an even positive integer, let D be a t-(v, k, λ) design, and for B a block of D, define $I(D, B)$ to be the set of all i for which some block of D, other than B, meets B in exactly i points. Now suppose that for some block B of D, $I(D, B)$ is contained in a set of t consecutive integers.

Then for every t-(v, k, λ) design E, every block C of E, and every $i = 0, \ldots, k$, the number of blocks of E meeting C in exactly i points is the same as the number of blocks of D meeting B in exactly i points.

In some sense, this result is best possible, for consider the 2-$(8, 4, 3)$ design Z given at the beginning of this talk.

The sizes of the intersections of the block 1234 with the other blocks of Z are the three consecutive integers $1, 2, 3$, and the sizes of the intersections of the block 1357 with the other blocks of Z are the two nonconsecutive integers $0, 2$.

For details, generalizations, proofs, and computer implementations, see:

P.J. Cameron and L.H. Soicher, Block intersection polynomials, *Bull. London Math. Soc.* **39** (2007), 559–564.

L.H. Soicher, More on block intersection polynomials and new applications to graphs and block designs, available from

`http://designtheory.org/library/preprints/`

L.H. Soicher, The DESIGN package for GAP, Version 1.3, 2006,

`http://designtheory.org/software/gap_design/`