Simplices in set systems

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A d-simplex is a collection of $d+1$ sets with empty intersection, every d of which have nonempty intersection. Several classical results in combinatorics can be phrased in terms of finding simplices in set systems on a given set of vertices: the Erdős-Ko-Rado theorem gives the largest k-uniform set system with no 1-simplex; the Ruzsa-Szemerédi $(6,3)$-theorem bounds the size of a simple triple system with no triangle (i.e. 2-simplex); Mantel's theorem gives the largest triangle-free graph.

In 1974 Chvátal posed the question of determining the largest k-uniform set system on n vertices with no d-simplex (Erdős had earlier asked the case $d = 2$). He conjectured that for $k \geq d+1$ and $n > k(d+1)/d$ the largest such system is a star, i.e. consists of all sets containing some specified point. This was proved by Frankl and Füredi in the case when n is extremely large as a function of k and d, but remains open in general. We give a proof of the conjecture for a range of parameters where n and k are linearly related. This also allows us to solve the non-uniform problem, generalising a question of Erdős and a result of Milner, showing that, for n sufficiently large, the unique largest set system on a set of n vertices that does not contain a d-simplex consists of all sets that either contain some specific element or have size at most $d-1$.

This is joint work with Dhruv Mubayi.