# Partition backtrack methods for more complicated group actions

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## 1 Outline

- **Combinatorial isomorphism** problems: action of group $G$ on set $\Omega$.
- **Current implementation** requires $G = S_n$.
- Not even possible to use an **arbitrary permutation group**.
- **Two goals**
  - **Augment** the framework for added flexibility.
  - **Generalise** the framework to work with general actions.
- **add** bells and whistles vs. **generalise** existing bells and whistles.
- **Progress** has been made on both.
- This talk: **generalisation**.

## 2 Generalisation: what, why, and how

### 2.1 What?

- **Two clear ways:**
  - Generalise the types of objects used to define permutations.
  - Generalise the type of group that acts on the objects.

- **Progress** has been made on the **latter.**
- Work is **very new** and certainly incomplete.
- When finished should provide a **"shopping list"** of functions.

## 2.2 Why?

- **Why generalise** to groups other than $S_n$?
  - **At least** deal with different permutation groups.
  - **Ideally** deal with abstract groups.
- **E.g.:** Different actions on the set $\mathbb{F}_q^{[n]}$ of length $n$ codewords.
  1. Act on $\mathbb{F}_q^{[n]}$ with $S_n$:
  $$\begin{aligned} S_n \times \mathbb{F}_q^{[n]} &\to \mathbb{F}_q^{[n]} \\ (x, \omega) &\mapsto \omega \circ x^{-1} \end{aligned}$$
  2. Act on $\mathbb{F}_q^{[n]}$ with $\mathbb{F}_q^\times \operatorname{wr} S_n$:
  $$\begin{aligned} \left(\left(F_q^\times\right)^{[n]} \times S_n\right) \times \mathbb{F}_q^{[n]} &\to \mathbb{F}_q^{[n]} \\ ((u, x), \omega) &\mapsto u \cdot (\omega \circ x^{-1}). \end{aligned}$$
  3. Act on $\mathbb{F}_q^{[n]}$ with $\operatorname{Aut}(\mathbb{F}_q) \times \left(\mathbb{F}_q^\times \operatorname{wr} S_n\right)$:
  $$\begin{aligned} \left(\operatorname{Aut}(\mathbb{F}_q) \times \left(\left(F_q^\times\right)^{[n]} \times S_n\right)\right) \times \mathbb{F}_q^{[n]} &\to \mathbb{F}_q^{[n]} \\ ((\lambda, (u, x)), \omega) &\mapsto \lambda(u \cdot (\omega \circ x^{-1})) \end{aligned}$$
- The **second and third** actions on $\mathbb{F}_q^{[n]}$ certainly don't seem isomorphic to symmetric groups.
- $\therefore$ current algorithm **falls short** of **naturally** working with sophisticated group actions.

## 2.3 How?

- Many mechanisms **dependent** on ordered partitions.
- $\therefore$ Want to keep using them.
- **But:** To move past using $S_n$ we need to move past $[n]$.
- Only concept dependent on $[n]$: the **group action**.
- All else depends on concepts that **generalise readily**.

# 3 Catalogue what we have

## 3.1 Objects

- Set of objects $\Omega$.
- $S_n$ **acts** on left of $\Omega$.

## 3.2 Points

- Set of points $[n]$.
- $S_n$ **acts faithfully** on left of $[n]$.

## 3.3 Partitions on points

- Set $\Pi_n$ of **ordered partitions** over $[n]$.
- $S_n$ **acts pointwise** on the left of $\Pi_n$.
- **Two critical aspects**.
- **Firstly:** each $\pi$ in $\Pi_n$ **defines subset** of $S_n$.
  - All perms that send $\pi$ to a **canonical representative** $h(\pi)$.
  - Obtained with function

$$\begin{aligned} \mathcal{B} : \Pi_n &\to \Pi_n \\ \pi &\mapsto \{\, x \in S_n \mid x\pi = h(\pi) \,\} \end{aligned}$$

  - **Harmonious partition** is canonical representative.

- **Secondly:** we construct a **refinement process**.
  - Gives a set of fine partitions for each $(\alpha, \pi) \in \Omega \times \Pi_n$.
  - Each **fine partition** gives **exactly one permutation**.
  - $\therefore$ a set of permutations is found.

- Properties of $\mathcal{B}$ and refinement $\Rightarrow$ **all automorphisms** and **canonical representative** obtained.

- What are these properties?

# 4 Distill the essentials

- **Two fundamental actions** of $S_n$: one on $\Omega$ and one on $[n]$.

- All other $G$ actions are **built up**.

- **Roughly:** Studying the (**probably complicated**) action of $S_n$ on $\Omega$ by working with the (**hopefully simpler**) action of $S_n$ over $[n]$.

- **Two stages** (concurrent in practice).

- **Firstly:** Generate (refine) ordered partitions.
  - **Critical**: All **functions** involved in **refinement** are $S_n$-morphisms.
  - Split:
$$\mathcal{S} : \Pi_n \times [n] \to \Pi_n.$$
  - Choose:
$$\mathcal{C} : \Omega \times \Pi_n \backslash \Phi_n \to \mathcal{P}\left([n]\right).$$
  - Refine:
$$\mathcal{R} : \Omega \times \Pi_n \to \Pi_n.$$
  - Tree:
$$\mathcal{T} : \Omega \times \Pi_n \to \mathcal{P}\left(\Omega \times \Pi_n\right).$$

- **Secondly:** Generate corresponding permutations.

- **Critical**: Function $h$ is a **canonical map** over $\Pi_n$ under $S_n$.
- **Practical need**: easily derive $h(\pi)$ and $\mathcal{B}(\pi)$.
- **Result**: Leaf permutation function

$$\mathcal{L} : \mathcal{P}(\Omega \times \Pi_n) \to \mathcal{P}(S_n)$$

is an $S_n$-morphism.

# 5 Generalise

## 5.1 $G$-spaces

- Act with an arbitrary group $G$.
- $\Omega$ is a $G$-space.
- Instead of $[n]$ use **finite set** $\Gamma$ forming a left $G$-space.
- Resulting set of ordered partitions: $\Pi_\Gamma$.
- Fine partitions: $\Phi_\Gamma$.
- $\Pi_\Gamma$ forms a left $G$-space under pointwise action.

## 5.2 Defining the subsets of $G$

- Analogous to $S_n$ on $\Pi_n$.
- **Essential:** Canonical map

$$\mu : \Pi_\Gamma \to \Pi_\Gamma.$$

- (What used to be $h$)
- Recall: $\forall \pi \in \Pi_\Gamma$, $\forall g \in G$

$$\mu(\pi) = \mu(g\pi)$$

and $\exists g' \in G$ s.t.

$$\mu(\pi) = g'\pi.$$

- Want elements of $G$ that send $\pi$ to its **canonical representative** $\mu(\pi)$.
- **Defn:**

$$\begin{aligned}\mathcal{B} : \Pi_\Gamma &\to \mathcal{P}\left(G\right)\\ \pi &\mapsto \{\, g \in G \mid g\pi = \mu(\pi) \,\}\end{aligned}$$

- **Prop:** $(\forall \pi \in \Pi_\Gamma)\ (\forall g \in \mathcal{B}\left(\pi\right))$

$$\mathcal{B}\left(\pi\right) = g\mathrm{Aut}\left(\pi\right)$$

- **Corr:** $\forall \pi \in \Pi_\Gamma$, $\forall g \in G$

$$\mathcal{B}\left(g\pi\right) = \mathcal{B}\left(\pi\right) g^{-1}$$

- So $\mathcal{B}$ is a $G$-morphism.
- **Only one** group element from a **fine** partition?
- **Corr:** Let $\pi$ be fine. $|\mathcal{B}\left(\pi\right)| = 1$ if and only if $G$ acts faithfully on $\Gamma$.
- **Proof:**
  - **Definition** $G$ is faithful on $\Gamma \iff \cap_{\gamma \in \Gamma}\mathrm{Aut}\left(\gamma\right) = \{1\}$.
  - **Since** $\pi$ **is fine** $\mathrm{Aut}\left(\pi\right) = \cap_{\gamma \in \Gamma}\mathrm{Aut}\left(\gamma\right)$.
  - **By Prop** $|\mathcal{B}\left(\pi\right)| = 1 \iff \mathrm{Aut}\left(\pi\right) = \{1\}$.
- **Tree function** gives a subset of $\Omega \times \Pi_\Gamma$.
- Need to **derive** the **group elements** corresponding to fine partitions.
- **Defn:**

$$\begin{aligned}\mathcal{L} : \mathcal{P}\left(\Omega \times \Pi_\Gamma\right) &\to \mathcal{P}\left(G\right)\\ A &\mapsto \{\, g \in G \mid (\exists (\alpha, \pi) \in A)\ \pi \in \Phi_\Gamma \wedge g \in \mathcal{B}\left(\pi\right) \,\}\end{aligned}$$

- **Prop:** $\mathcal{L}$ is a $G$-morphism; that is, $(\forall A \in \mathcal{P}\left(\Omega \times \Pi_\Gamma\right))\ (\forall g \in G)$

$$\mathcal{L}\left(gA\right) = \mathcal{L}\left(A\right) g^{-1}.$$

## 5.3 Refining partitions of $\Pi_\Gamma$

- **Refinement relation** $\sqsubseteq$ can still be defined.
- **Same approach**: split, choose, and refine.
  - Split:
  $$\mathcal{S} : \Pi_\Gamma \times \Gamma \to \Pi_\Gamma.$$
  - Choose:
  $$\mathcal{C} : \Omega \times \Pi_\Gamma \backslash \Phi_\Gamma \to \mathcal{P}(\Gamma).$$
  - Refine:
  $$\mathcal{R} : \Omega \times \Pi_\Gamma \to \Pi_\Gamma.$$
- All **domains and codomains** are $G$-spaces
- We require all these functions to be $G$**-morphisms**.
- Definition of the **tree function** is identical to before.
- However,
$$\mathcal{T} : \Omega \times \Pi_\Gamma \to \mathcal{P}(\Omega \times \Pi_\Gamma)$$
- Induction is still useable; therefore,
  - $\mathcal{T}$ is a $G$-morphism
  - $\mathcal{T}$ always gives a set with at least one fine partition.

## 5.4 Combining the two

- $\mathcal{L} \circ \mathcal{T}$ will be a $G$-morphism.
- **Prop:**
$(\forall (\alpha, \pi) \in \Omega \times \Pi_\Gamma)\ (\forall g \in G)$
$$(\mathcal{L} \circ \mathcal{T}(g\alpha, g\pi))(g\alpha, g\pi) = (\mathcal{L} \circ \mathcal{T}(\alpha, \pi))(\alpha, \pi).$$
- **Prop:**
$(\forall (\alpha, \pi) \in \Omega \times \Pi_\Gamma)\ (\forall g \in \mathcal{L} \circ \mathcal{T}(\alpha, \pi))$
$$g\mathrm{Aut}((\alpha, \pi)) \subseteq \mathcal{L} \circ \mathcal{T}(\alpha, \pi).$$

## 6 Outstanding tasks

- Still need to get at a **base and strong generating set**.
- **May impose** a few extra **limitations** on what function $\mu$ can be.
- Nice interaction with the refinement relation may be one such need.
- **Examples** of functions that fit the shopping list are needed.