Graph homomorphisms II: some examples

Taoyang Wu
Combinatorics Study Group Notes 2, October 2006

October 6, 2006

Abstract

Following the talk on graph homomorphisms given by Peter last week, we continue to discuss some examples of graph homomorphisms. More precisely, the graph parameters which can be represented by counting the graph homomorphisms. The main reference is Section 2 in [2].

1 Introduction

In this note, we will study some of explicit examples about graph homomorphism, which provides a useful language and motivation to continue our study about[2]. The basic setting is as follows: $G = (V(G), E(G))$ is a simple graph unless stated otherwise, $\phi : G \to H$ is a homomorphism from G to H and $hom(G, H)$ is the number of homomorphisms from G to H.

In fact, we should consider the homomorphisms both "from" G and "to" G. The basic scheme in the paper [2] is:

$$F \to G \to H.$$

Given a (large, simple) graph G, we can study its local structure by counting of various "small" graphs F into G; and its global structure by counting its homomorphisms into various small graph H. Roughly speaking, we can get some information about G via "probing from the left with F", which is related to property testing. On the other hand, we can "probing from the right with H, which is related to statistical physics. Informally, H is also called the template (Model) of G. One useful observation is that any $\phi : G \to H$ gives a partition on $V(G)$ via the fibres of ϕ.

2 Weighted and unweighted

A *weighted graph* H is a graph with a positive real weight $\alpha_H(i)$ associated with each node i and a real weight $\beta_H(i,j)$ associated with each edge ij.

An edge with weight 0 will play the same as role as no edge between those nodes, so we could assume that we only consider weighted complete graphs with loops at all nodes. An unweighted graph is a weighted graph where all the nodeweights and edgeweights are 1.

Let G and H be two weighted graphs. To every map $\phi : V(G) \to V(H)$, we assign the weight:

$$hom_\phi(G,H) = \prod_{uv\in E(G)} [\beta_{\phi(u)\phi(v)}(H)]^{\beta_{uv}(G)} \tag{1}$$

here $(0^0 = 1)$. We then define

$$hom(G,H) = \sum_{\phi:V(G)\to V(H)} \alpha_\phi hom_\phi(G,H) \tag{2}$$

where

$$\alpha_\phi = \prod_{u\in V(G)} [\alpha_{\phi(u)}(H)]^{\alpha_u(G)}. \tag{3}$$

We'll use this definition most often in the case when G is a simple unweighted graph, so that:

$$\alpha_\phi = \prod_{u\in V(G)} [\alpha_{\phi(u)}(H)].$$

and

$$hom_\phi(G,H) = \prod_{uv\in E(G)} [\beta_{\phi(u)\phi(v)}(H)]$$

3 Example

In this section we will present some examples about $hom(F,G)$ ($hom(G,H)$), where F (H) is a single graph or a subfamily of finite graphs.

3.1 Left & unweighted

Example 1. Let F be a point, then $hom(F,G) = |V(G)|$.
Let F be K_2, then $hom(F,G) = 2|E(G)|$.
Let F be K_3, then $hom(F,G) = 6 \times \#$ of triangles.(note: here G is assume to be simple.)

Example 2. Let F be a t-path P_k, then $hom(F, G)$=# of walks with length t. Here (i, j, k) and (k, j, i) should be treated as two different walks of length 3.

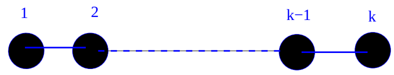

Figure 1: the path on k nodes

Example 3. Let S_k be the star on k nodes, then $hom(S_k, G) = \sum_{i=1}^n d_i^{k-1}$ where d_i is the degree of $i \in V(G)$. Hence $hom(S_k, G)^{1/(k-1)}$ tends to the maximum degree of G as $k \to \infty$.

Proof. Denote the vertices set of S_k by $\{1, 2, \cdots, k\}$. Pick any $v \in V(G)$, and study the number of homomorphisms of $\phi : S_k \to G$ s.t $\phi(1) = v$. For each vertex of S_k other than 1, there are d_v different choices in $V(G)$ as its image where d_v is the degree of d_v. Therefore we have totally d_v^{k-1} such homomorphisms. On the other hand, the image of 1 can run through all vertices of G, therefore the number of homomorphisms between S_k and G is $\sum_{i=1}^n d_i^{k-1}$.

□

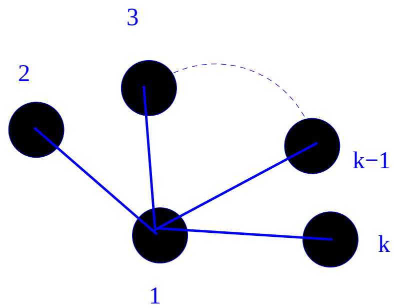

Figure 2: the star on k nodes

Example 4. Let C_k be the cycles on k nodes, then $hom(F, G) = \sum_{i=1}^n \lambda_i^k$ where λ_i is the eigenvalue of G.

Proof. In this example, G is not necessarily to be simple. Firstly we claim the number of loops in G is equal to the sum of its eigenvalues. That is because this sum is the trace of A_G, the adjacent matrix of G, and we know the trace of A_G counts the number of loops in G. Secondly we note that

$\sum_{i=1}^{n} \lambda_i^k$ is the trace of A_{G^k} where $V(G^k) = V(G)$ and $(i,j) \in E(G^k)$ iff there is exactly one path of length k from i to j in the graph G. More precisely, $A_{G^k} = \underbrace{A(G) \times A(G) \cdots \times A(G)}_{k}$. On the other hand, the number of loops in G^k is the same as $hom(C_k, G)$, which complete the proof. □

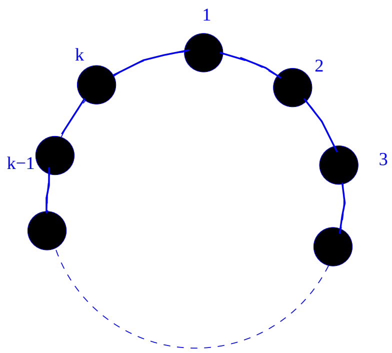

Figure 3: the cycles on k nodes

Example 5. (Random graphs) Let $G(n,p)$ be a random graph with n nodes and edgedensity p. Then for every simple graph F with k nodes,

$$E(hom(F,G)) = (1+o(1))n^k p^{|E(F)|} \quad (n \to \infty).$$

Proof. Given any map ϕ from F to G, denote the probability that ϕ is a homomorphism by ρ. Then we know $\rho = p^{|E(F)|}$ when ϕ is 1-1 and $p^{|E(F)|} \leq \rho \leq 1$ otherwise. On the other hand, the number of the 1-1 map from F to G is $n(n-1)\cdots(n-k+1)$ while the total number of the maps is n^k. Put all these together, we have:

$$E(hom(F,G)) = n(n-1)\cdots(n-k+1)p^{|E(F)|} + (n^k - n(n-1)\cdots(n-k+1))t$$

where $p^{|E(F)|} \leq t \leq 1$. Let $\delta = t - p^{|E(F)|}$. Then $0 \leq \delta \leq 1$ and we have:

$$\begin{aligned} E(hom(F,G)) &= n^k p^{|E(F)|} + (n^k - n(n-1)\cdots(n-k+1))\delta \\ &= (1+o(1))n^k p^{|E(F)|} \quad (n \to \infty) \end{aligned}$$

since

$$\lim_{n\to\infty} \frac{(n^k - n(n-1)\cdots(n-k+1))\delta}{n^k p^{|E(F)|}} = 0$$

for fixed p, k, $E(F)$ and $0 \leq \delta \leq 1$. □

3.2 Right & unweighted

Example 6. Let H be a point, then $hom(G, H) \neq 0$ iff G has no edge.
Let H be K_2, then $hom(G, H) \neq 0$ iff G is bipartite.
Let H be K_3, then $hom(G, H) \neq 0$ iff G is 3-colorable.

Example 7. (Independent Set) Let H be the graph on two nodes, with an edge connecting the two nodes and a loop at one of the nodes. Then $hom(G, H)$ is the number of independent sets of nodes in G.

Proof. The independent set is 1-1 corresponding to $\phi^{-1}(2)$. More precisely, given any independent set A, there is a unique $\phi : G \to H$ such that $A = \phi^{-1}(2)$. On the other hand, $\phi^{-1}(2)$ is an independent set for any given homomorphism $\phi : G \to H$. □

Note: If H has only two nodes $\{1, 2\}$, any $\phi : G \to H$ is uniquely decided by any fibre of ϕ. ($\phi^{-1}(1)$ or $\phi^{-1}(2)$) Or we can say such ϕ given a partition on G. In this sense we can there exists a 1-1 corresponding between the partitions and the homomorphisms where (V_1, V_2) and (V_2, V_1) are two different partitions. In the above example, the part of partition corresponding to $\phi^{-1}(2)$ is exactly the independent set, which is implied by the fact that point 2 is loopless.

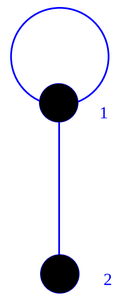

Figure 4: the target for Independent Set

Example 8. (Colorings) Let H be K_t, then $hom(G, H) = \#$ of the colorings of the graph G with t colors.

Note:We can consider the homomorphisms into a fixed graph H as generalized colorings, called the H colorings. There are also other generalization, such as circular colorings.

3.3 Weighted examples

Example 9. (Maximum cut) Let H denote the looped complete graph on two nodes, weighted as follows: the non-loop edge has eight 2; all other edges and nodes have weight 1. Then for every simple graph G with n nodes,

$$\log_2 hom(G, H) - n \leq \text{MaxCut}(G) \leq \log_2 hom(G, H).$$

where $\text{MaxCut}(G)$ denotes the size of the maximum cut in G. So unless G is very sparse, $\log_2 hom(G, H)$ is a good approximation of the maximum cut in G.

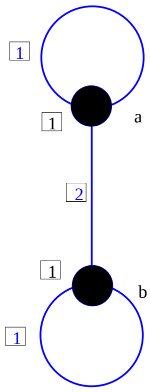

Figure 5: the target for Maximum Cut

Proof. If $\{V_1, V_2\}$ is a partition of $V(G)$, the set $E(V_1, V_2)$ of all edges of G crossing this partition is called a cut, whose size is denoted by $t_{\{V_1,V_2\}}$. From the discussion in Example 7, we known there is $1-1$ corresponding between the partitions and the homomorphisms. On the other hand, the corresponding between partitions and cuts is many to one.

Choose a cut of maximal size, denote by t_{max}, and a homomorphism ϕ_{max} (not unique) corresponding to this cut.

From the weight on H, we know:

$$hom(G, H) = \sum_{\phi: G \to H} hom_\phi(G, H) = \sum_{\phi: G \to H} 2^{t_\phi}$$

where t_ϕ is the cut size corresponding to the partition given by ϕ.

Then we have

$$2^{t_{\max}} = hom_{\phi_{max}}(G, H) \leq hom(G, H),$$

which means

$$\text{MaxCut}(G) \leq \log_2 hom(G, H).$$

On the other hand,

$$hom(G, H) \leq 2^n hom_{\phi_{max}}(G, H) = 2^{n+t_{max}}$$

as there are total 2^n homomorphisms from G to H, which implies:

$$\log_2 hom(G, H) - n \leq \text{MaxCut}(G).$$

□

Example 10. (Partition functions of the Ising model) Let G be any simple graph, and let $T > 0$, $h \geq 0$, and J be three real parameters. Let H be the looped complete graph on two nodes, denoted by $+$ and $-$, weighted as follows: $\alpha_+ = e^{h/T}$, $\alpha_- = e^{-h/T}$, $\beta_{++} = \beta_{--}$, and $\beta_{++}/\beta_{+-} = e^{2J/T}$. Then $hom(G, H)$ is the paritition function of the Ising model on the graph G at temperature T with coupling J in external magnetic field h.

Proof. The proof will be left as an exercise to the reader. (Hint: the configurations is 1-1 corresponding to the homomorphims via its fibres.)

□

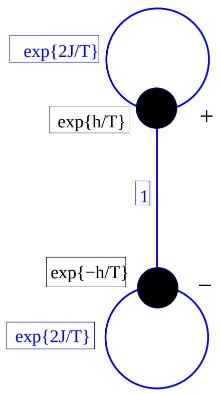

Figure 6: the target for partition function

3.4 As a language

- Chromatic Number:

$$\chi(G) = \min_k \{k \mid hom(G, K_n) \neq 0\}$$

- Clique Number:

$$\omega(G) = \max_n \{k \mid hom(K_n, G) \neq 0\}$$

- Odd girth:

$$og(G) = \min_l \{2l+1 \mid hom(C_{2l+1}, G) \neq 0\}$$

4 A nontrivial example

4.1 Nowhere-zero flows

Let Γ be a finite abelian group and let S be a subset of Γ s.t S is closed under inversion.

For any graph G, fix an orientation of the edges. An *S-flow* is an assignment of an element of S to each edge s.t for each node v, the sum of elements assigned to edges entering v is the same as the sum of elements assigned to edges leaving v.

Let $f(G)$ be the number of S-flows. This number is independent of the orientation.

Let $S = \Gamma \setminus \{0\}$. Then such an S-flow is called nowhere-zero flows.

A Eulerian tour is an S-flow when $\Gamma = \mathbb{Z}_2$ and $S = \mathbb{Z}_2 \setminus \{0\}$.

4.2 The representation of flows number

Let H be the complete directed graph (with all loops) on $\hat{\Gamma}$. Let $\alpha_\chi \triangleq \frac{a}{|\Gamma|}$ for each $\chi \in \Gamma$, and let

$$\beta_{\chi,\chi'} \triangleq \sum_{s \in S} \chi^{-1}(s)\chi'(s),$$

for $\chi, \chi' \in \hat{\Gamma}$. Then $f(G)$ can be described as a homomorphism function [1].

Theorem 4.1. *$f(G) = hom(G, H)$.*

The proof of this theorem is rather technical and will be put in the appendix. Instead, we will present two examples: the first one is the H for the Eulerian characteristic function and the second is the H for the Nowhere-zero 4 flows.

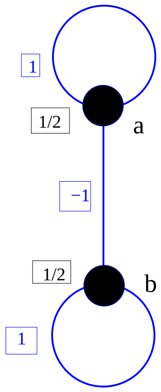

Figure 7: The H for Eulerian

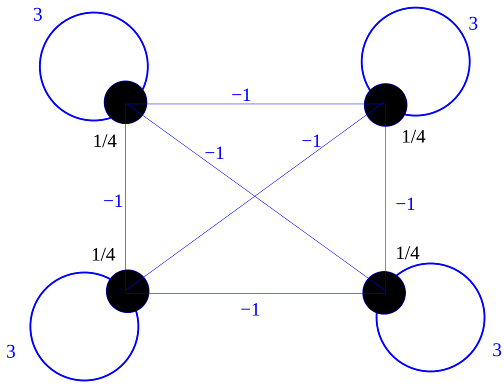

Figure 8: the H for Nowhere-zero 4 flow

5 Some elementary properties

If F is the disjoint union of two graphs F_1 and F_2, then

$$hom(F, G) = hom(F_1, G)hom(F_2, G).$$

If F is connected and G is the disjoint union of two graphs G_1 and G_2, then

$$hom(F, G) = hom(F, G_1) + hom(F, G_2).$$

Thus in a sense it's enough to study homomorphisms between connected graphs.

For two simple graphs G_1, G_2, their *categorial product* $G_1 \times G_2$ is defined to be a graph with vertices set $V(G_1) \times V(G_2)$, in which (i_1, j_1) is connected to (i_1, j_2) iff $(i_1, i_2) \in E(G_1)$ and $(j_1, j_2) \in E(G_2)$. For this product, we have:

$$hom(F, G_1 \times G_2) = hom(F, G_1) \cdot hom(F, G_2)$$

6 Conclusion

Today we are focused on some examples of graph parameters which can be represented by counting the graph homomorphisms. A natural question would be: what are the parameters that are unable to have such representation? Surprisingly, the authors of [2] obtain some exact conditions to character such parameters, which would likely to be the topic of the next talk of this series. Or the reader can consult Section 3 in [2].

7 Appendix

7.1 The Ising model

Let $G = (V, E)$ be a finite graph, and call $\Omega = \{-1, +1\}^V$ the state space, with elements $\sigma = \{\sigma_x\}_{x\in V}$. The varaiable $\sigma_x \in \{-1, +1\}$ is called the spin at vertex x. This is a *spin system.*

There is an *energy function* defined on Ω. For the Ising model this function is defined as:

$$H(\sigma) = -J \sum_{(x,y)\in E} \sigma_x \sigma_y - \sum_{x\in V} h\sigma_x.$$

where J is a real constant, the *interaction strength*, and $h \in \mathbb{R}$, decided by an *external magnetic field.*

Now we introduce the *inverse temperature* parameter $\beta \backsim \frac{1}{T}$,and consider the following probability measure on Ω:

$$\mu(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_\beta}, \qquad Z_\beta = \sum_{\sigma\in\Omega} e^{-\beta H(\sigma)}. \tag{4}$$

This Z_β is called the *partition function*, and, as usually generating fuctions do, contains basically all information about the systems.

7.2 Group characters

A *character* χ of Γ is a homomorphism $\Gamma \to \mathbb{S}^1$ where $\mathbb{S}^1$ is the multiplicative group of complex numbers of modulus 1. The *unit character* χ_0 is the character which assigns 1 to every element in Γ.

We list a few facts about characters:

- A function $\chi : \Gamma \to \mathbb{C}$ is a character iff it satisfies $\chi(a + b) = \chi(a) + \chi(b), a, b \in \Gamma$ since Γ is finite.
- The set of all characters of Γ form a group $\hat{\Gamma}$, called the dual group of Γ.
- $\hat{\Gamma} \cong \Gamma$.

Proposition 7.1.

$$\sum_{\chi\in\hat{\Gamma}} \chi(a) = \begin{cases} n & \textit{if } a = 0 \\ 0 & \textit{otherwise} \end{cases}$$

7.3 The proof of Theorem 4.1

Proof. Let $n = |V(G)|$ and $m = |\Gamma|$.

For any coloring $\psi : E(G) \to S$ and node $v \in V(G)$, let

$$\partial_\psi(v) = \sum_{u \in V(G)\ uv \in E(G)} \psi(uv) - \sum_{u \in V(G)\ vu \in E(G)} \psi(vu).$$

So ψ is an S-flow iff $\partial_\psi = 0$.

Consider the expression

$$A = \sum_{\psi : E(G) \to S} \prod_{v \in V(G)} \sum_{\chi \in \hat{\Gamma}} \chi(\partial_\psi(v)). \tag{5}$$

Form Proposition 7.1, the summation over χ is 0 unless $\partial_\psi(v) = 0$, in which case it is m. So the product over $v \in V(G)$ is 0 unless ψ is an S-flow, in which case is it is m^n. Therefore $A \cdot m^{-n}$ counts S-flows.

On the other hand, we can expand the product over $v \in V(G)$; this step looks like:

$$\begin{array}{rcr} & (\chi_0(\partial_\psi(v_1)) + \chi_1(\partial_\psi(v_1)) + \cdots + \chi_{m-1}(\partial_\psi(v_1))) & \times \\ & (\chi_0(\partial_\psi(v_2)) + \chi_1(\partial_\psi(v_2)) + \cdots + \chi_{m-1}(\partial_\psi(v_2))) & \times \\ & \cdots & \times \\ & (\chi_0(\partial_\psi(v_n)) + \chi_1(\partial_\psi(v_n)) + \cdots + \chi_{m-1}(\partial_\psi(v_n))) & \\ = & \sum_\phi (\chi_{\phi(v_1)}(\partial_\psi(v_1)))(\chi_{\phi(v_2)}(\partial_\psi(v_2))) \cdots (\chi_{\phi(v_n)}(\partial_\psi(v_n))) & \end{array}$$

Each term in the sum is corresponding to a choice of a character χ_{ϕ_v} for each v. Denote χ_{ϕ_v} by ϕ_v and so we get

$$A = \sum_{\psi : E(G) \to S} \sum_{\phi : V(G) \to \hat{\Gamma}} \prod_{v \in V(G)} \phi_v(\partial_\psi(v)).$$

Here(using that ϕ_v is a character)

$$\phi_v(\partial_\psi(v)) = \prod_{u \in V(G)\ uv \in E(G)} \phi_v(\psi(uv)) \prod_{u \in V(G)\ vu \in E(G)} \phi_v(\psi(vu))^{-1}$$

So we get that

$$A = \sum_{\psi : E(G) \to S} \sum_{\phi : V(G) \to \hat{\Gamma}} \prod_{uv \in E(G)} \phi_v(\psi(uv))\ \phi_u(\psi(uv))^{-1}.$$

Interchanging the summation, we have:

$$\begin{aligned} A &= \sum_{\phi:V(G)\to\hat{\Gamma}} \sum_{\psi:E(G)\to S} \prod_{uv\in E(G)} \phi_v(\psi(uv))\ \phi_u(\psi(uv))^{-1} \\ &= \sum_{\phi:V(G)\to\hat{\Gamma}} \prod_{uv\in E(G)} \sum_{s\in S} \phi_v(s)\phi_u(s)^{-1} \\ &= \sum_{\phi:V(G)\to\hat{\Gamma}} \prod_{uv\in E(G)} \beta_{\phi(u),\phi(v)} \\ &= \sum_{\phi:V(G)\to\hat{\Gamma}} m^n hom_\phi(G,H) \\ &= m^n hom(G,H). \end{aligned}$$

where in the last two steps we use the following fact:

$$\begin{aligned} hom(G,H) &= \sum_{\phi:V(G)\to V(H)} (\alpha_\phi)(hom_\phi(G,H)) \\ &= \sum_{\phi:V(G)\to V(H)} (\prod_{u\in V(G)} [\alpha_{\phi(u)}(H)])(\prod_{uv\in E(G)} [\beta_{\phi(u)\phi(v)}(H)]) \\ &= \sum_{\phi:V(G)\to V(H)} (\frac{1}{m^n})(\prod_{uv\in E(G)} [\beta_{\phi(u)\phi(v)}(H)]). \\ &= \sum_{\phi:V(G)\to V(H)} (\frac{1}{m^n}) hom_\phi(G,H). \end{aligned}$$

Therefore, we have:

$$f(G) = m^{-n}A = hom(G,H)$$

which completes our proof. □

References

[1] M.Freedman, L.Lovasz, A. Schrijver, Reflection positivity, rank connectivity, and homomorphism of graphs.

[2] Ch.Borgs, J.Chayes, L.Lovasz, V.T.Sos and K.Vesztergombi, couting graph homomorphisms.