Graphs and finite transformation monoids

pjc, January 2009

This note describes a pair of mappings between graphs and transformation monoids on the set $\{1,\ldots,n\}$, and some of their properties.

A *homomorphism* from a graph X to a graph Y is a map from the vertex set of X to that of Y which maps edges to edges; its behaviour on non-edges is unspecified. An *endomorphism* of X is a homomorphism from X to X. The set endomorphisms of a graph X, of course, is closed under composition and contains the identity; that is, it forms a monoid $\mathrm{End}(X)$. An endomorphism is an *automorphism* if it is bijective. The automorphisms of X form a permutation group (a subgroup of the symmetric group S_n), the *automorphism group* of X, denoted $\mathrm{Aut}(X)$.

Two graphs X and Y are *hom-equivalent* if there are homomorphisms in both directions between them. A graph X is a *core* if there is no graph on fewer vertices which is hom-equivalent to X. Every graph X is hom-equivalent to a unique core, called the *core of* X, and written $\mathrm{Core}(X)$. The core of X has an embedding into X, and there is a retraction (a homomorphism which is the identity on its image) from X to the image of the embedding. We can recognise a core by the following property:

Fact 1 *The graph X is a core if and only if* $\mathrm{End}(X) = \mathrm{Aut}(X)$.

The *clique number* $\omega(X)$ of X is the size of the largest complete subgraph of X; the *chromatic number* $\chi(X)$ is the smallest number of colours required for a proper colouring of X. A class of graphs which will be very important in the sequel are those having the properties of the following result:

Fact 2 *The graph X has the property that the core is complete if and only if* $\omega(X) = \chi(X)$.

Proof $\omega(X) = m$ if and only if there is a homomorphism $K_m \to X$, while $\chi(X) = m$ if and only if there is a homomorphism $X \to K_m$. So $\omega(X) = \chi(X) = m$ if and only if X is homomorphically equivalent to K_m (which is then necessarily the core of X).

In the other direction, let M be a transformation monoid on $\{1,\dots,n\}$, a submonoid of the full transformation monoid T_n. From M, we construct a graph as follows. Its vertex set is $\{1,\dots,n\}$; two vertices i and j are joined by an edge if and only if there is no element $f \in M$ for which $if = jf$. Denote this graph by $\mathrm{Gr}(M)$.

A transformation monoid is *synchronizing* if it contains an element whose image has cardinality 1.

Fact 3 (a) $\mathrm{Gr}(M)$ *is complete if and only if M is a permutation group (that is, contained in the symmetric group).*

(b) $\mathrm{Gr}(M)$ *is null if and only if M is synchronizing.*

Proof (a) $\mathrm{Gr}(M)$ is complete if and only if no element of M ever maps two points to the same place.

(b) Let $f \in M$ be an element whose image is as small as possible. Then no two elements of the image of f can be mapped to the same place; so they are pairwise adjacent. So, if $\mathrm{Gr}(M)$ is null, then the image of f has cardinality 1. The converse is clear.

Fact 4 *For any transformation monoid M, the graph* $\mathrm{Gr}(M)$ *has core a complete graph.*

Proof The argument in (b) above shows that the image of an element of M of minimal rank is a complete subgraph of $\mathrm{Gr}(M)$. It is hom-equivalent to $\mathrm{Gr}(M)$ (the homomorphism in the other direction is just the embedding), and it is clearly a core.

Fact 5 *For any transformation monoid M,*

(a) $M \le \mathrm{End}(\mathrm{Gr}(M))$;

(b) $\mathrm{Gr}(\mathrm{End}(\mathrm{Gr}(M))) = \mathrm{Gr}(M)$.

Proof (a) Let f be an endomorphism of M, and let i and j be adjacent in $\mathrm{Gr}(M)$. By definition, $if \neq jf$. Could if and jf be non-adjacent in $\mathrm{Gr}(M)$? if so, then there is an element $h \in \mathrm{End}(M)$ with $(if)h = jf(h)$. But this contradicts the adjacency of i and j, since $fh \in M$ by closure.

(b) Suppose first that i and j are adjacent in $\mathrm{Gr}(M)$. Then no endomorphism of $\mathrm{Gr}(M)$ can collapse them, so they are adjacent in $\mathrm{Gr}(\mathrm{End}(\mathrm{Gr}(M)))$.

Conversely, suppose that i and j are not adjacent in $\mathrm{Gr}(M)$. Then there is an element $f \in M$ satisfying $if = jf$. By (a), $f \in \mathrm{End}(\mathrm{Gr}(M))$, and so i and j are non-adjacent in $\mathrm{Gr}(\mathrm{End}(\mathrm{Gr}(M)))$.

It is not true that $\mathrm{End}(\mathrm{Gr}(\mathrm{End}(X))) = \mathrm{End}(X)$ for all graphs X. For let X be the path of length 3, with just two automorphisms. It is easy to see that no endomorphism can identify the ends of the path, so that $\mathrm{Gr}(\mathrm{End}(X))$ is the 4-cycle, with eight automorphisms.

Fact 6 *The maps $M \mapsto \mathrm{End}(\mathrm{Gr}(M))$ and $X \mapsto \mathrm{Gr}(\mathrm{End}(X))$ are idempotent.*

Proof This follows immediately from part (b) of the preceding Fact.

Write $\mathrm{Cl}(M) = \mathrm{End}(\mathrm{Gr}(M))$. Then $M \leq \mathrm{Cl}(M)$ and $\mathrm{Cl}(\mathrm{Cl}(M)) = \mathrm{Cl}(M)$, so Cl is a closure operator on transformation monoids on $\{1,\ldots,n\}$. I don't have a satisfactory description of the closed objects; but more on this below.

In the other direction, let $\mathrm{Hull}(X) = \mathrm{Gr}(\mathrm{End}(X))$, so that $\mathrm{Hull}(\mathrm{Hull}(X)) = \mathrm{Hull}(X)$. The hull of a graph has the following properties:

Fact 7 (a) *X is a spanning subgraph of $\mathrm{Hull}(X)$ (that is, these graphs have the same vertex set, and every edge of X is an edge of $\mathrm{Hull}(X)$).*

(b) $\mathrm{End}(X) \leq \mathrm{End}(\mathrm{Hull}(X))$ *and* $\mathrm{Aut}(X) \leq \mathrm{Aut}(\mathrm{Hull}(X))$.

(c) $\mathrm{Core}(\mathrm{Hull}(X))$ *is a complete graph on the vertex set of* $\mathrm{Core}(X)$.

Proof (a) If i and j are adjacent in X, then no endomorphism of X can collapse i and j, so they are adjacent in $\mathrm{Gr}(\mathrm{End}(X))$.

(b) Immediate from Fact 5(a).

(c) The vertex set of $\mathrm{Core}(X)$ cannot be collapsed by endomorphisms, so is a complete subgraph of $\mathrm{Gr}(\mathrm{End}(X)) = \mathrm{Hull}(X)$.

By (c), if X is a hull, then $\mathrm{Core}(X)$ is complete; but the converse is false. If X is the path of length 3, then $\mathrm{Core}(X)$ is a complete graph on two vertices, but $\mathrm{Hull}(X)$ is the 4-cycle, by our previous argument.

Fact 8 *A transformation monoid M is closed (that is, satisfies $M = \mathrm{Cl}(M)$) if and only if $M = \mathrm{End}(X)$ for some graph X which is a hull (and in particular, whose core is complete).*

Proof Suppose that M is closed. Then $M = \mathrm{End}(X)$, where $X = \mathrm{Gr}(M)$; so $X = \mathrm{Gr}(\mathrm{End}(X)) = \mathrm{Hull}(X)$.

Conversely, if $X = \mathrm{Hull}(X)$, then $\mathrm{End}(X) = \mathrm{End}(\mathrm{Gr}(\mathrm{End}(X))) = \mathrm{Cl}(\mathrm{End}(X))$.