A New Proof of Pappus's Theorem

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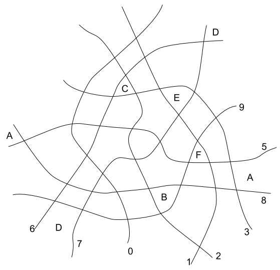
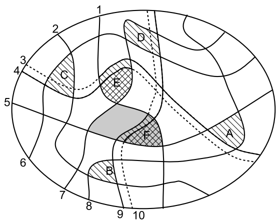
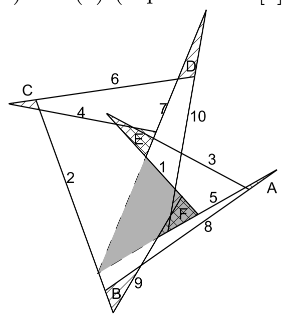


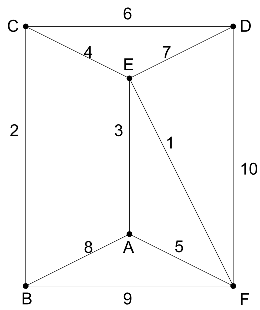
(a):$\mathbf{Rin}(9)$ (copied from [4])



(b): (a) projected



(c): (b) partially stretched



(d): underlying graph of (c)

All labels correspond to (d). Note $(c\prime)$ below is part of (c).

To reduce space we write S_i^j for $\sin(\theta_j - \theta_i)$.

1. MAIN THEOREM

The realizability problem for rank 3 oriented matroids (see [1]) is equivalent to the pseudoline stretchability problem (see [4]). This paper uses an example to illustrate a new approach to this problem.

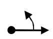
The main theorem of this paper, like the trigonometric form of Ceva's theorem, shows a non-trivial relationship amongst the angles in a specific line arrangement figure (c). We work in polar coordinates, with the origin always in the open region of the Euclidean plane, between the last and the first lines (under the polar orientation). To describe those parts of (c) that are significant (the nine shaded triangles, and the polar origin $(c\prime)$) we use the notation: $(i, j, k)^+$ to show that the lines labeled i and k cross on the far side of line j, and $(i, j, k)^-$ to show that they cross on the same side of j as the origin. The use of either statement also indicates that $0 < \theta_j - \theta_i, \theta_k - \theta_j, \theta_k - \theta_i < 180$.

Theorem 1. *In any arrangement of 10 lines with polar coordinates* $\{(r_i, \theta_i) : i = 1, \dots 10\}$, *with the nine following oriented triangles* $(3, 5, 8)^-$, $(2, 8, 9)^-$, $(2, 4, 6)^+$, $(6, 7, 10)^-$, $(1, 3, 7)^-$, $(1, 4, 7)^-$, $(1, 5, 9)^+$, $(1, 5, 10)^+$, $(1, 5, 7)^+$, *we have:*

$$S_8^9 S_1^{10} S_2^4 S_3^5 S_6^7 + S_4^6 S_1^9 S_7^{10} S_3^5 S_2^8 - S_4^6 S_3^8 S_7^{10} S_2^9 S_1^5 > 0 \qquad (1)$$

2. RELATIONSHIP WITH PAPPUS

Ringel [7] showed that any stretched version of the projective pseudoline arrangement $\mathbf{Rin}(9)$ shown in (a) would contradict Pappus' theorem. The Euclidean pseudoline arrangement (b) is derived from (a) by taking line 0 as the line at infinity, and adding lines 4 and 10, parallel to lines 3 and 9 respectively. Any stretching of $\mathbf{Rin}(9)$ would provide a Euclidean stretching of (b), necessarily satisfying the premises of the main theorem. In addition, we would have that $\theta_4 = \theta_3$, $\theta_{10} = \theta_9$, and $0 < \theta_j - \theta_i < 180$ for all $i < j$. Repeated application of the identity $\sin(\alpha-\gamma)\sin(\beta-\delta) = \sin(\alpha-\beta)\sin(\gamma-\delta) + \sin(\alpha-\delta)\sin(\beta-\gamma)$ to the expression in 1, gives a sum of negative terms, giving a contradiction. Reversing Ringel's argument then provides the new proof of Pappus theorem.



$(c\prime)$ Origin for (c)

3. STRICT LINEAR INEQUALITIES

Each triangle $(i, j, k)^+$ corresponds to the strict inequality $r_i S_j^k - r_j S_j^k + r_k S_i^j > 0$, see [2]. Thus we can form the following linear system in $\boldsymbol{r}$.

$$\begin{pmatrix}
 & & -\boldsymbol{S_5^8} & & S_3^8 & & & -\boldsymbol{S_3^5} & & \\
 & -\boldsymbol{S_8^9} & & & & & & \boldsymbol{S_2^9} & -\boldsymbol{S_2^8} & \\
 & \boldsymbol{S_4^6} & & -\boldsymbol{S_2^6} & & \boldsymbol{S_2^4} & & & & \\
 & & & & & -\boldsymbol{S_7^{10}} & S_6^{10} & & & -\boldsymbol{S_6^7} \\
-S_3^7 & & \boldsymbol{S_1^7} & & & & -S_1^3 & & & \\
-S_4^7 & & & \boldsymbol{S_1^7} & & & -S_1^4 & & & \\
S_5^9 & & & & -S_1^9 & & & & \boldsymbol{S_1^5} & \\
S_5^{10} & & & & -S_1^{10} & & & & & \boldsymbol{S_1^5} \\
S_5^7 & & & & -S_1^7 & & & & S_1^5 &
\end{pmatrix} \boldsymbol{r} > \boldsymbol{0} \qquad (2)$$

The 8 by 7 submatrix shown in bold, is a simplex, in the terms of Motzkin's analysis [5] of such systems. That there is a single linear dependency between the rows of this system is shown by consideration of the graph, (d), underlying (c) (by a 'twisting' process, that turns vertices of G into triangles of (c), with each line in (c) having at least one such triangle on each side). In particular, the graph G is three edge connected, and has an induced connected cubic subgraph H, such for every vertex v of G, has at least $\deg(v) - 2$ neighbours in H, which is a sufficient condition for systems derived from G in this way, to contain such a simplex. Using Motzkin's theorem D3, further general analysis of possible simplices in the full matrix, including the last row, proves the main theorem.

4. ON ORIENTED MATROID THEORY

In [1], p348, a different proof of the non-realizability of $\mathbf{Rin}(9)$ is given, using the method of final polynomials. While that method both extends beyond rank 3 and has been found generally useful, it is silent about the realizability of an oriented matroid for which no final polynomial is found. In contrast, this method, was initially developed for pseudoline stretching ([3]): approximating real linear systems such as 2, by rational linear systems, soluble using linear programming. It is hoped that, when completed, it will provide a uniform framework for either finding a non-realizability proof, as in this paper, or for providing a realization.

5. REFERENCES

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