Problems from the First Anglo-Hungarian Groups and Geometry meeting

This list is quite informal. I have not attributed problems (except to the presenter) or provided references. Some of the problems are well-known. If anyone wants to make additions or corrections, please let me know.

Symmetric graph designs

A *symmetric graph design*, or SGD, with parameters (n, X, λ, F), (where n and λ are positive integers and X and F are graphs on n vertices), is a set $\{X_1, \ldots, X_n\}$ of subgraphs of the complete graph on $\{1, \ldots, n\}$ having the following properties:

(a) $X_i \cong X$ for $i = 1, \ldots, n$;

(b) $X_i \cap X_j \cong F$ for $i, j = 1, \ldots, n$, $i \neq j$;

(c) any edge of K_n lies in exactly λ of the graphs $X_1, \ldots, X_n$.

To avoid degenerate cases, we assume that X is neither the complete nor the null graph on n vertices. Symmetric graph designs generalise symmetric BIBDs (symmetric 2-designs, the case where $X = K_k$, $F = K_\lambda$). In this case, condition (b) follows from conditions (a) and (c).

Problem: Does condition (b) follow from (a) and (c) in general? Does (c) follow from (a) and (b)?

Peter Cameron

Weakly associative lattices

Let X be a set and $\leq$ a binary relation on X. Assume that

(a) $\leq$ is reflexive and antisymmetric;

(b) for $x \neq y$, there are exactly λ common upper bounds and λ common lower bounds of x and y, with $\lambda > 1$.

Let $U(x) = \{y \in X : x \leq y\}$. Then $(X, \{U(x) : x \in X\})$ is a symmetric 2-design (BIBD).

Conversely, if $(X, \mathcal{B})$ is a symmetric design, and $\phi : X \to \mathcal{B}$ is a function satisfying

(c) $x \in \phi(x)$ for all x,

(d) for $x \neq y$, $x \in \phi(y)$ implies $y \notin \phi(x)$,

and we set $x \leq y$ if $y \in \phi(x)$, then (a) and (b) hold.

Examples can be obtained from difference sets. For example, if q is a prime power congruent to 3 mod 4, and we take $X = \mathrm{GF}(q)$ and set $x \leq y$ if $y - x$ is a square, then we obtain such a structure.

Problem: Find other examples.

Andras Gács

Reconstruction of words

Let $\sim_k$ be the relation on words over the alphabet $\{a, b\}$ defined as follows: $w_1 \sim_k w_2$ if every word of length at most k has the same number of occurrences in w_1 and w_2 as a (not necessarily consecutive) subword. Thus, for example, $abba \sim_2 baab$.

Problem: Given k, find the smallest n for which there exist distinct words w_1, w_2 of length N satisfying $w_1 \sim_k w_2$.

Peter Sziklai

Covering radius

Given a matrix A over a (finite) field F, there are two structures associated with A:

(a) the matroid (representable over F) whose elements are the columns of A, in which independence means linear independence;

(b) the linear code spanned by the rows of A.

It was proved by Curtis Greene that the weight enumerator of the code is a specialisation of the Tutte polynomial of the code.

The *covering radius* of a code $C \subseteq F^n$ is the least number r such that every element of F^n lies at distance r or smaller from some word in C. We have examples of codes with the same weight enumerator but different covering radii.

Problem: Do there exist codes whose associated matroids have the same Tutte polynomial but which have different covering radii?

Carrie Rutherford

A representation group

Consider the following geometry. Let V be a vector space of dimension $2n$ over GF(2) carrying a symplectic form. The points of the geometry are the totally isotropic subspaces of dimension n; the lines are the totally isotropic subspaces of dimension $n-1$; and a point and line are incident if one contains the other. Then any line has three points. Let P be the set of points.

Let M_n be the elementary abelian 2-group with presentation

$$\begin{aligned} M_n = \langle a_x (x \in P) : \quad & a_x^2 = 1, [a_x, a_y] = 1, \\ & a_x a_y a_z = 1 \text{ for each line } \{x, y, z\} \rangle. \end{aligned}$$

Problem: Prove the conjecture of Brouwer that

$$\dim M_n = 1 + \begin{bmatrix} n \\ 1 \end{bmatrix}_2 + \begin{bmatrix} n \\ 2 \end{bmatrix}_2 ,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the number of k-dimensional subspaces of $\mathrm{GF}(q)^n$.

Sasha Ivanov

Finitary groups

Let G be a subgroup of $\mathrm{FGL}(V)$, where V is an infinite-dimensional vector space over a division ring, and $\mathrm{FGL}(V)$ is the group of all *finitary* linear transformations g of V, those for which the image of $g-1$ is finite-dimensional. Suppose that G is torsion-free, irreducible (fixes no proper subspace of V), and primitive (fixes no non-trivial direct decomposition of V).

Problem: Prove the conjecture of Phillips that G has a free normal subgroup.

Chris Pinnock

Generating finite simple groups

Any finite simple group S can be generated by two elements. Let $m(S)$ be the cardinality of the largest subset of S any two of whose elements generate S. How large is $m(S)$?

In the case $S = \mathrm{PSL}(d, q)$, let $m'(S)$ be the cardinality of the largest set of Singer elements of S (that is, elements which act irreducibly on the underlying vector space) any two of which generate S. Is $m'(S) = m(S)$? If not, is $m'(S) \sim m(S)$ as $|S| \to \infty$?

Aron Bereczky