Solutions to Exercises
Chapter 3: Subsets, partitions, permutations

1 A restaurant near Vancouver offered Dutch pancakes with 'a thousand and one combinations' of toppings. What do you conclude?

Since $1001 = \binom{14}{4}$, it is plausible that there were fourteen different toppings, of which any four could be chosen. (In fact, 1001 simply meant a large number; there were 25 different toppings, any subset being permitted.)

2 Using the numbering of subsets of $\{0, 1, \ldots, n-1\}$ defined in Section 3.1, prove that, if $X_k \subseteq X_l$, then $k \leq l$ (but not conversely).

Let $(a_0, \ldots, a_{n-1})$ and $(b_0, \ldots, b_{n-1})$ be the characteristic functions of X_k and X_l (the binary digits of k and l respectively). Each of a_i and b_i is either zero or one. If $X_k \subseteq X_l$, then it is impossible that $a_i = 1$ and $b_i = 0$; hence $a_i \leq b_i$ for all i, and so

$$k = \sum a_i 2^i \leq \sum b_i 2^i = l.$$

The converse is false: $1 < 2$, but $X_1 = \{0\}$ is not a subset of $X_2 = \{1\}$.

3 Prove that, for fixed n, the greatest binomial coefficient $\binom{n}{k}$ occurs when $k = \lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$.

The ratio of successive binomial coefficients is given by $\binom{n}{k+1} / \binom{n}{k} = (n-k)/(k+1)$. Now, if $k < (n-1)/2$, then $n + k > k + 1$, and this ratio is greater than 1, so the binomial coefficients increase. If n is odd, then the ratio is equal to 1 for $k = (n-1)/2$, so that $\binom{n}{(n-1)/2} = \binom{n}{(n+1)/2}$; after that, the ratio is less than 1 and the coefficients decrease again. So the two middle coefficients are the largest. If n is even, the ratio is never equal to 1, and so the largest binomial coefficient is the first for which the ratio is less than 1, namely $\binom{n}{n/2}$.

4 Prove the following identities:

(a) $\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l}$.

(b) $\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$

(recall the convention that $\binom{n}{k} = 0$ if $k < 0$ or $k > n$).

(c) $\sum_{i=0}^{k}\binom{n+i}{i} = \binom{n+k+1}{k}$.

(d) $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

(e) $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} 0 & \text{if } k \text{ is odd;} \\ (-1)^m\binom{2m}{m} & \text{if } n = 2m. \end{cases}$

(a) Let X be an n-set. Count pairs (Y,Z) with $Z \subseteq Y \subseteq X$ and $|Y| = k$, $|Z| = l$. There are $\binom{n}{k}$ choices for Y, and then $\binom{k}{l}$ choices of an l-subset Z of Y. Alternately, there are $\binom{n}{l}$ choices of Z; then we obtain Y by choosing $k-l$ points from $X \setminus Z$ to adjoin to Z, and since $|X \setminus Z| = n-l$, this can be done in $\binom{n-l}{k-l}$ ways.

If $k < l$, then no choices are possible, and both sides are zero, as a result of our conventions about binomial coefficients.

(b) Choose a team of k players from a class with m girls and n boys.

(c) The result can be proved by induction, being clear when $k = 0$. Assuming it for $k-1$, we have

$$\sum_{i=0}^{k}\binom{n+i}{i} = \binom{n+k}{k-1} + \binom{n+k}{k} = \binom{n+k+1}{k},$$

and the induction step is proved.

(d) Differentiating the Binomial Theorem gives $\sum_{k=1}^{n} k\binom{n}{k}t^{k-1} = n(1+t)^{n-1}$ (the term with $k = 0$ is constant and its derivative is zero). Now put $t = 1$.

(e) Consider the identity $(1+t)^n(1-t)^n = (1-t^2)^n$, and calculate the coefficient of t^n. On the left, we multiply the coefficient of t^{n-k} in $(1+t)^n$ (which is $\binom{n}{n-k} = \binom{n}{k}$) by the coefficient of t^k in $(1-t)^n$ (which is $(-1)^k\binom{n}{k}$), and sum over

k. On the right, it is zero if n is odd (since only even powers of t appear), and $(-1)^{n/2}\binom{n}{n/2}$ if n is even, as required.

5 Following the method in the text, calculate the number of subsets of an n-set of size congruent to $m \pmod 3$ ($m = 0,1,2$) for each value of $n \pmod 6$.

Let $\alpha = e^{2\pi i/3}$ and $\beta = e^{4\pi i/3}$ be the two cube roots of unity. Then $1+\alpha+\beta = 0$. We have

$$\begin{aligned} \sum_{k=0}^{n}\binom{n}{k} &= (1+1)^n = 2^n, \\ \sum_{k=0}^{n}\binom{n}{k}\alpha^k &= (1+\alpha)^n = (-\beta)^n, \\ \sum_{k=0}^{n}\binom{n}{k}\beta^k &= (1+\beta)^n = (-\alpha)^n. \end{aligned}$$

The values of $(-\beta)^n$ are $1, -\beta, \alpha, -1, \beta, -\alpha$ according as $n \equiv 0,1,2,3,4,5 \pmod 6$. Also, $1+\alpha^k+\beta^k = 3$ if $k \equiv 0 \pmod 3$, and 0 otherwise. Adding the three equations shows that the number of subsets of size divisible by 3 is equal to $(2^n+(-\beta)^n+(-\alpha)^n)/3$, which is $(2^n+2)/3$, $(2^n+1)/3$, $(2^n-1)/3$, $(2^n-2)/3$, $(2^n-1)/3$, or $(2^n+1)/3$, depending on the congruence of n modulo 6.

For sets of size congruent to 1 mod 3, multiply the second equation by α^2 and the third by β^2 before adding; for 2 mod 3, use the multipliers α and β instead.

6 Let k be a given positive integer. Show that any non-negative integer N can be written uniquely in the form

$$N = \binom{x_k}{k} + \binom{x_{k-1}}{k-1} + \ldots + \binom{x_1}{1},$$

where $0 \le x_1 < \ldots < x_{k-1} < x_k$. [HINT: Let x be such that $\binom{x}{k} \le N < \binom{x+1}{k}$. Then any possible representation has $x_k = x$. Now use induction and the fact that $N - \binom{x}{k} < \binom{x}{k-1}$ (Fact 3.2.5) to show the existence and uniqueness of the representation.]

Show that the order of k-subsets corresponding in this way to the usual order of the natural numbers is the same as the reverse lexicographic order generated by the algorithm in Section 3.11. [HINT: $\sum_{j=0}^{i}\binom{n-j}{i-j} = \binom{n+1}{i}$.]

Given N and k, choose x such that $\binom{x}{k} \le N < \binom{x+1}{k}$. We must have $x_k \le x$. Moreover,

$$\sum_{i=1}^{k} \binom{x_i}{i} \le \sum_{i=1}^{k} \binom{x_k - k + i}{i} \le \binom{x_k + 1}{k} - 1,$$

(the first inequality holding since $x_i \le x_k - k + i$ and the second by Exercise 3(c)); so $x_k < x$ would be impossible. Thus we must have $x_k = x$. By induction on k, the integer $N - \binom{x}{k}$ has a unique expression in the same form with $k-1$ replacing k; and, as in the hint, Fact 3.2.5 shows that $x_{k-1} < x_k$. So the result is proved.

The last part is proved by induction; the inductive step involves considering the algorithm (3.12.3). Suppose that, in the expression for N, we have $x_i = n + i$ for $i = 1, \ldots, r-1$, and $x_r > n + r$. Then $\sum_{i=1}^{r-1} \binom{x_i}{i} = \binom{n+r}{r} - 1$, by Exercise 3(c) (note that the term $i = 0$ is missing from the sum). So, if we increase x_{r-1} by 1 and set $x_i = i - 1$ for $i < r - 1$, the corresponding sum is increased by 1, and represents $N + 1$. So the k-set representing $N + 1$ is the same as the next one produced by (3.12.3), that is, the next in reverse lexicographic order.

7 Use the fact that $(1+t)^p \equiv 1 + t^p \pmod{p}$ to prove by induction that $n^p \equiv n \pmod{p}$ for all positive integers n.

Clearly $1^p \equiv 1 \pmod{p}$. Assuming that the result is true for n, we have

$$(1+n)^p \equiv 1 + n^p \equiv 1 + n \pmod{p},$$

the first congruence by the given fact and the second by the inductive hypothesis. So the inductive step holds.

The given fact is proved on page 28.

8 A computer is to be used to calculate values of binomial coefficients. The largest integer which can be handled by the computer is 32767. Four possible methods are proposed:

(1) $\binom{n}{k} = n!/k!(n-k)!$;

(2) $\binom{n}{k} = n(n-1)\dots(n-k+1)/k!$;

(3) $\binom{n}{0} = 1, \quad \binom{n}{k} = \binom{n}{k-1} \cdot \frac{n-k+1}{k}$ for $k > 0$;

(4) $\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $0 < k < n$ (i.e., Pascal's Triangle).

For which values of n and k can $\binom{n}{k}$ be calculated by each method? What can you say about the relative speed of the different methods?

Method 1 will fail if $n! > 32767$, that is, if $n > 7$; so it computes $\binom{n}{k}$ for $n \leq 7$.

Method 2 requires the numerator to be at most 32767; so, for given k, it works for n up to some value $n(k)$, where $n(k) = 32767$, 181, 33, 15, 10, 8, 7 for $k = 1, \dots, 7$.

Method 3 requires $\binom{n}{k}(n-k+1) \leq 32767$, which bounds n by a similar function $m(k)$; we find that $m(k) = 32767$, 181, 41, 22, … .

Method 4 reaches each binomial coefficient "from below", so it calculates all those $\binom{n}{k}$ which do not exceed 32767. For example, for $k = 2$, 3 it works up to $n = 256$, 59 respectively. (In addition, it is the only method which works for k close to n.)

Of the best two, methods 3 and 4, we see that 3 is inferior in terms of the range of values for which it works. It is faster for computing a single binomial coefficient, requiring only $k-1$ multiplications and $k-1$ divisions, whereas method 4 needs roughly nk additions. But method 4 is preferable if the entire Pascal triangle is required, since each addition yields a new piece of data.

9 Show that there are $(n-1)!$ cyclic permutations of a set of n points.

A cyclic permutation, written in cycle form, consists of the n points written in some order inside a bracket. There are $n!$ orders. However, the cycle can start at any point, and so n orders yield the same cyclic permutation. Thus, the number of cyclic permutations is $n!/n = (n-1)!$, as required.

10 The *order* of a permutation π is the least positive integer m such that π^m is the identity permutation. Prove that the order of a cycle on n points is n. Prove that the order of an arbitrary permutation is the least common multiple of the lengths of the cycles in its cycle decomposition.

If the point x lies in an n-cycle of π, then on successive applications of π, x visits in turn all the points of the cycle, returning to its starting point after n applications. So $x\pi^n = x$ and $x\pi^k \neq x$ for $0 < k < n$.

Thus, if π is a single n-cycle, then π^k is not the identity for $0 < k < n$; but π^n maps every point to itself, that is, π^n is the identity. So the order of π is n.

More generally, we see that $x\pi^k = x$ if and only if k is a multiple of the length of the cycle in which x lies. So, for an arbitrary permutation π, we see that π^k is the identity if and only if k is a multiple of the length of every cycle of π. So the order of π, being the least positive k for which this holds, is the least common multiple of the cycle lengths.

11 How many words can be made from the letters of the word ESTATE?

There are $24+24+12+4+1 = 65$ words with no repeated letters. If E but not T is repeated, there are $10\cdot 6+6\cdot 6+3\cdot 3+1 = 106$ words. (If there are $n-2$ further letters, then there are $\binom{n}{2}$ positions for the Es, and $(3)_{n-2}$ choices for the positions of the other letters, where $(n)_k = n(n-1)\cdots(n-k+1)$.) The same number occur if T but not E is repeated. If both pairs are repeated, there are $15\cdot 6\cdot 2+10\cdot 3\cdot 2+6 = 246$ words. (Choose the positions for the Es, then those for the Ts, then those for the remaining letters.) In total, 523 words.

12 Given n letters, of which m are identical and the rest are all distinct, find a formula for the number of words which can be made.

If we use k of the m identical letters and l of the others, there are $\binom{k+l}{k}(n-m)_l$ possible words. Now sum over k and l.

13 Show that, for $n = 2,3,4,5,6$, the number of *unlabelled* trees on n vertices is 1, 1, 2, 3, 6 respectively.

This is done by drawing the trees: for $n = 4$, the two trees are a path with three edges, or three edges radiating from a central vertex. As a check that nothing has been forgotten, the number of labellings of a tree is equal to $n!$ divided by the number of automorphisms (symmetries) of the tree. (So, for example, the 4-vertex path has two symmetries, the identity and reversal, and the other tree has six, since the three edges can be permuted arbitrarily; and we have $4!/2+4!/6 = 4^{4-2}$, in accordance with Cayley's Theorem.)

You can find drawings of the trees in N. J. A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, 1995, Figure M0791. The sequence is online here.

14 The line segments from $(i, \log i)$ to $(i+1, \log(i+1))$ lie below the curve $y = \log x$. (This is because the curve is convex, i.e., its second derivative $-1/x^2$ is negative.) The area under these line segments from $i = 1$ to $i = n$ is $\log n! + \frac{1}{2}\log(n+1)$, since it consists of the rectangles of Fig. 3.1(b) together with triangles with width 1 and heights summing to $\log(n+1)$. Deduce that

$$n! \leq \mathrm{e}\sqrt{n+1}\left(\frac{n}{\mathrm{e}}\right)^n.$$

This is mostly spelt out in the question. The area of the rectangles is $\sum_{i=2}^{n} \log i = \log n!$, while the triangles add up to half a rectangle of height $\log(n+1)$. So

$$\log n! + \tfrac{1}{2}\log(n+1) \leq \int_1^{n+1} \log x \, \mathrm{d}x = (n+1)\log(n+1) - (n+1) + 1,$$

giving $n! \leq \sqrt{n+1}((n+1)/\mathrm{e})^n$. The final result is obtained using the fact that $(n+1)^n \leq \mathrm{e}n^n$ (see Exercise 3(b) of Chapter 2).

15 Use Stirling's Formula to prove that

$$\binom{2n}{n} \sim 2^{2n}/\sqrt{\pi n}.$$

By Stirling's formula,

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{4\pi n}(2n/\mathrm{e})^{2n}}{2\pi n(n/\mathrm{e})^2 n} = \frac{2^{2n}}{\sqrt{\pi n}}.$$

(Check the definition of the relation $\sim$ to ensure that this is valid.)

16 (a) Let $n = 2k$ be even, and X a set of n elements. Define a *factor* to be a partition of X into k sets of size 2. Show that the number of factors is equal to $1 \cdot 3 \cdot 5 \cdots (2k-1)$. This number is sometimes called a *double factorial*, written $(2k-1)!!$ (with !! regarded as a single symbol, the two exclamation marks suggesting the gap of two, *not* the factorial function iterated!)

(b) Show that a permutation of X interchanges *some* k-subset with its complement if and only if all its cycles have even length. Prove that the number of such permutations is $((2k-1)!!)^2$.

(c) Deduce that the probability that a random element of S_n interchanges some $\frac{1}{2}n$-set with its complement is $O(1/\sqrt{n})$.

We get a 1-factor by writing $n/2$ boxes each with room for two entries, and filling them with the elements $1, \ldots, n$. These can be written in the boxes in $n!$ ways. However, permuting the boxes (in $k!$ ways), or the elements within the boxes (in 2^k ways) doesn't change the 1-factor. The product of these numbers is $2.4.6 \ldots (2k)$; dividing, we obtain $1.3.5 \ldots (2k-1) = (2k-1)!!$ for the number of 1-factors.

(b) Suppose that the k-set A is exchanged with its complement B by a permutation. Then elements of A and B alternate around each cycle, which thus has even length. Conversely, if all cycles have even length, we may colour the elements in each cycle alternately red and blue; the red and blue sets are then exchanged.

From a permutation with all cycles even we obtain a pair of 1-factors as follows. A 2-cycle is assigned to both 1-factors; in a longer cycle, the consecutive pairs are assigned alternately to the two 1-factors. The process isn't unique, since the starting point isn't specified; indeed, a permutation gives rise to 2^d ordered pairs of 1-factors, where d is the number of cycles of length greater than 2.

Conversely, let a pair of 1-factors be given. Their union is a graph with all vertices of valency 2, thus a disjoint union of circuits, all of even length (since the 1-factors alternate around a circuit). We take these circuits to be the cycles of a permutation. In fact, for a circuit of length greater than 2, there are two choices for the direction of traversal. So the number of permutations obtained is 2^d, where d is as before.

(c) The proportion is

$$\begin{aligned} ((2k-1)!!)^2/(2k)! &= \prod_{i=1}^{k} (1 - 1/2i) \\ &\leq \prod_{i=1}^{k} \mathrm{e}^{-1/2i} \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-\sum_{i=1}^{k}(1/2i)} \\
&= \mathrm{e}^{-\log k/2+O(1)} = O(k^{-1/2}).
\end{aligned}
$$

17 How many relations on an n-set are there? How many are (a) reflexive, (b) symmetric, (c) reflexive and symmetric, (d) reflexive and antisymmetric?

A relation is a set of ordered pairs, so the first part of the question asks for the number of subsets of a set of size n^2 (the number of ordered pairs).

(a) For a reflexive relation, we must include all pairs (x,x), together with an arbitrary subset of the pairs (x,y) with $x \neq y$; there are $n(n-1)$ unequal pairs.

(b) There are $n(n+1)/2$ unordered selections of two elements from n, with repetitions allowed. A symmetric relation is determined by a subset of these, since it must include both or neither of (x,y) and (y,x) for all x,y.

(c) We must put in all pairs (x,x), and decide about pairs (x,y) with $x \neq y$. So the relation is determined by a subset of the set of unordered selections without repetitions.

(d) Again we include all pairs (x,x). For each x,y with $x \neq y$, there are three possibilities: include (x,y), or (y,x), or neither.

18 Given a relation R on X, define

$$R^+ = \{(x,y) : (x,y) \in R \text{ or } x = y\}.$$

Prove that the map $R \to R^+$ is a bijection between the irreflexive, antisymmetric and transitive relations on X, and the reflexive, antisymmetric and transitive relations on X. Show further that this bijection preserves the property of trichotomy.

Let R be irreflexive, antisymmetric and transitive. We claim first that R^+ is reflexive, antisymmetric and transitive. The first two assertions are clear; we must verify the transitive law. So suppose that $(x,y),(y,z) \in R^+$. If $(x,y),(y,z) \in R$ then $(x,z) \in R$ by the transitivity of R; if $(x,y) \in R$ and $y = z$ then $(x,z) \in R$; and the other two cases are similar.

Conversely, let S be reflexive, antisymmetric and transitive, and let

$$S^- = \{(x,y) : (x,y) \in S \text{ and } x \neq y\}.$$

Clearly S^- is irreflexive and antisymmetric. We show that it is transitive. Suppose that $(x,y),(y,z) \in S^-$. Then $(x,y),(y,z) \in S$, so $(x,z) \in S$, by transitivity of S. Could we have $x = z$? No, since then $(x,y),(y,x) \in S$ with $x \neq y$, contradicting antisymmetry. So $(x,z) \in S^-$.

Moreover, $(R^+)^- = R$ and $(S^-)^+ = S$, so we have a bijection as claimed.

For the final part, we must show that R^+ satisfies trichotomy if and only if R does. This is clear: for trichotomy can be expressed as "if $x \neq y$, then one of (x,y) and (y,x) satisfies the relation", and the pairs (x,y) with $x \neq y$ are the same in R and R^+.

19 Recall that a *partial preorder* is a relation R on X which is reflexive and transitive. Let R be a partial preorder. Define a relation S by the rule that $(x,y) \in S$ if and only if both (x,y) and (y,x) belong to R. Prove that S is an equivalence relation. Show further that R 'induces' a partial order $\overline{R}$ on the set of equivalence classes of S in a natural way: if $(x,y) \in R$, then $(\bar{x},\bar{y}) \in \overline{R}$, where $\bar{x}$ is the S-equivalence class containing x, etc. (You should verify that this definition is independent of the choice of representatives x and y.)

Conversely, let X be a set carrying a partition, and R' a partial order on the parts of the partition. Prove that there is a unique partial preorder on X giving rise to this partition and partial order as in the first part of the question.

Show further that the results of this question remain valid if we replace *partial preorder* and *partial order* by *preorder* and *order* respectively, where a *preorder* is a partial preorder satisfying trichotomy.

Let R be a partial preorder (a reflexive and transitive relation), and let

$$S = \{(x,y) : (x,y) \in R \text{ and } (y,x) \in R\}.$$

Now

- S is reflexive: for $(x,x) \in R$ for all x.
- S is symmetric, by definition.
- S is transitive: for, given $(x,y),(y,z) \in S$, we have $(x,y),(y,z) \in R$ so $(x,z) \in R$, and also $(y,x),(z,y) \in R$ so $(z,x) \in R$, whence $(x,z) \in S$.

Thus S is an equivalence relation.

Let $\bar{x}$ be the equivalence class containing x. We put

$$\overline{R} = \{(\bar{x},\bar{y}) : (x,y) \in R\}.$$

Note that the definition is independent of the choice of representatives of the equivalence classes. For, if $x' \in \bar{x}$ and $y' \in \bar{y}$, and $(x,y) \in R$, then $(x',x),(x,y),(y,y') \in R$, so $(x',y') \in R$. Now the verification that $\overline{R}$ is reflexive and transitive is straightforward. To show that $\overline{R}$ is antisymmetric, suppose that $(\bar{x},\bar{y}),(\bar{y},\bar{x}) \in \overline{R}$. Then $(x,y),(y,x) \in R$, and so $(x,y) \in S$, whence $\bar{x} = \bar{y}$.

The verification that trichotomy carries over from R to $\overline{R}$ is "more of the same".

20 List the (a) partial preorders, (b) preorders, (c) partial orders, (d) orders on the set $\{1,2,3\}$.

The numbers are (a) 29, (b) 13, (c) 19, (d) 6. (The numbers of unlabelled structures are 9, 4, 5, 1 respectively.)

It is not too much labour to draw all the unlabelled orders, etc., and count the numbers of labellings of each. For partial orders on up to four points, see N. J. A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, 1995, Figure M1495, online here.

21 Prove that $B_n < n!$ for all $n > 2$.

With every permutation of a set, there is associated the partition of the set into disjoint cycles of the permutation (see page 30). Every partition arises from a permutation in this way: take a cyclic permutation of each part of the partition, and compose them. So the number $n!$ of permutations is at least as great as the number B_n of partitions. The partition with a single part is associated with the cyclic permutations, of which there are $(n-1)!$ (Exercise 8); this number is greater than 1 if $n \geq 3$, so $B_n < n!$ for $n > 2$.

22 Verify, theoretically or practically, the following algorithm for generating all partial permutations of $\{1,\ldots,n\}$:

(3.13.1) Algorithm: Partial permutations of $\{1,\ldots,n\}$

- FIRST PARTIAL PERMUTATION is the empty sequence.
- NEXT OBJECT after $(x_1,\ldots,x_m)$:
  - If the length m of the current sequence is less than n, extend it by adjoining the least element it doesn't contain.
  - Otherwise, proceed as in the algorithm for permutations, up to the point where x_j and x_k are interchanged; then, instead of reversing the terms after x_j, remove them from the sequence.

Use induction on n: check by hand that the algorithm is correct for small n. (For example, when $n = 3$, it generates in turn $\emptyset$, 1, 12, 123, 13, 132, 2, 21, 213, 23, 231, 3, 31, 312, 32, 321, and then terminates.) When it reaches i, it then generates in order all partial permutations which begin with i (since the same algorithm is being applied to the last $n-1$ positions using the symbols different from i), and then proceeds to $i+1$.

23 Verify the following recursive procedure for generating the set of partitions of a set X.

(3.13.2) Recursive algorithm: Partitions of X

- If $X = \emptyset$, then $\emptyset$ is the only partition.
- If $X \neq \emptyset$, then
  - select an element $x \in X$;
  - generate all subsets of $X \setminus \{x\}$;
  - for each subset Y, generate all partitions of $X \setminus (\{x\} \cup Y)$, and adjoin to each the additional part $\{x\} \cup Y$.

This algorithm is just the constructive form of the proof of the recurrence (3.12.1).

24 Let $A = (a_{ij})$ and $B = (b_{ij})$ be $(n+1) \times (n+1)$ matrices (with rows and columns indexed from 0 to n) defined by $a_{ij} = \binom{i}{j}$, $b_{ij} = (-1)^{i+j}\binom{i}{j}$ (where $\binom{i}{j} = 0$ if $i < j$). Prove that $B = A^{-1}$.

The vector $(x_0, \ldots, x_n)$ represents (with respect to the basis consisting of powers of t) the polynomial $f(t) = \sum x_i t^i$. Now let $f(t+1) = \sum x_i (t+1)^i = \sum y_i t^i$. By the Binomial Theorem, $y_j = \sum_i \binom{i}{j} x_j$; so the transformation $f(t) \mapsto f(t+1)$ is represented by the matrix A. Similarly, the transformation $f(t) \mapsto f(t-1)$ is represented by B. But these transformations are obviously inverses of each other.

A direct proof would run as follows. We are required to show that

$$\sum_j \binom{i}{j}\binom{j}{k}(-1)^{j-k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}.$$

Now $\binom{i}{j} = 0$ unless $j \leq i$, and $\binom{j}{k} = 0$ unless $k \leq j$; so the sum is zero if $i > k$, and is one if $i = k$ (since then only the term $j = i = k$ is non-zero). Try to prove that the sum is zero for $i < k$. (Your conclusion may well be that this is harder than the conceptual proof outlined above.)

No solutions will be given for the projects. A crib for 25 is I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration*, John Wiley & Sons, New York, 1986. In 27, the limiting ratio was shown by A. Rényi to be $\sqrt{\mathrm{e}}$: see *Publ. Math. Inst. Hungar. Acad. Sci.* **4** (1959), 73–85.