Problems from the Twelfth British Combinatorial Conference

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Abstract

These problems were presented at the twelfth British Combinatorial Conference, held at the University of East Anglia in 1989. I have updated the references and hope to append comments on the subsequent history of the problems.

Problem 1 (BCC12.1) Some doubly resolvable designs. Proposed by R. A. Bailey.
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For which n and k does there exist a doubly resolvable incomplete block design for nk treatments in n^2 blocks of size k, so that all concurrences are 0 or 1?

Such a design exists if there are k MOLS of order n. What about other values? In particular, does it exist for $(n,k) = (6,3)$, $(6,4)$, $(6,5)$ or $(10,3)$?

Problem 2 (BCC12.2) Latin squares with transitive groups.
Proposed by R. A. Bailey.

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Do there exist any Latin squares, other than Cayley tables, whose automorphism groups are transitive?

Problem 3 (BCC12.3) Two Diophantine equations. Proposed by Peter Cameron.

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Let q, n, k be positive integers with $q > 1$, $n > 3$.

(a) Show that

$$\binom{k}{2} - 1 = \frac{q^n - 1}{q - 1}$$

has no solutions.

(b) Show that

$$\binom{k}{2} - 1 = q^n + 1$$

has only the solution $q^n = 64$, $k = 12$.

Remark: Richard Guy remarks that, for fixed n, these equations are presumably of the sort to which Faltings' theorem applies, in which case there are only finitely many solutions for each n. But surely these equations are less difficult than Fermat's last theorem!

Problem 4 (BCC12.4) Generalised quadrangles. Proposed by Peter Cameron.

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Show that there is no generalised quadrangle with parameters s, t, where s is finite and greater than 1, and t is infinite.

Note: A *generalised quadrangle* is a geometry of points and lines, with any two points on at most one line, such that if point P is not on line L, then P is collinear with a unique point of L. It has parameters s, t if each line contains $s+1$ points and each point lies on $t+1$ lines. If $s = 1$, it is a complete bipartite graph. The finiteness of t when $s = 2, 3$ was shown by Cameron and Kantor respectively.

Problem 5 (BCC12.5) Edge-density of infinite graphs. Proposed by Peter Cameron.

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Let G be an infinite graph. List all *labelled* n-vertex subgraphs of G (a finite list, for each n), and let d_n be the average density of edges in the list (that is, the total number of edges divided by $\binom{n}{2}$ times the number of graphs in the list. Does $\lim_{n\to\infty} d_n$ exist?

Problem 6 (BCC12.6) Intersecting families. Proposed by P. J. Cameron, P. Frankl and W. M. Kantor.

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Let $\mathcal{F} \subseteq \mathcal{P}(\{1, 2, \ldots, n\})$ be an intersecting family of sets which is maximal (that is, $|\mathcal{F}| = 2^{n-1}$) and regular (each point lies in the same number of elements of $\mathcal{F}$). Let m be the size of the smallest set in $\mathcal{F}$. It is known that $m \geq \frac{1}{2}\log n + c$; examples are known with $m \sim \sqrt{n}$. (See [4]). What is the truth?

Editor's Note: This problem has been solved by Aaron Meyerowitz [8], who showed that the lower bound is correct.

Problem 7 (BCC12.7) Forbidding divisibility. Proposed by P. J. Cameron and P. Erdős.

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Let $f(n)$ be the number of sequences

$$1 \leq a_1 < \ldots < a_t \leq n$$

with $a_i \not| \, a_j$ for $i \neq j$. Prove that $\lim_{n\to\infty} f(n)^{1/n}$ exists.

Note: The limit should be about 1.58: see [3].

Problem 8 (BCC12.8) Inverting sign matrices. Proposed by Ömer Eğecioğlu.

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Consider two disjoint finite sets A and B, each partitioned into n^2 not necessarily nonempty subsets A_{ij} and B_{ij} for $i, j = 1, \ldots, n$, respectively. Suppose that we have a mapping sign : $A \cup B \to \{-1, 1\}$. Let

$$a_{ij} = \sum_{x \in A_{ij}} \operatorname{sign}(x), \qquad b_{ij} = \sum_{y \in B_{ij}} \operatorname{sign}(y).$$

In many combinatorial situations, the numbers a_{ij} and b_{ij} are given not explicitly but *algorithmically.*

Put

$$I_{ij} = \bigcup_{k=1}^{n} A_{ik} \times B_{kj}, \qquad I = \bigcup_{i,j=1}^{n} I_{ij}.$$

Clearly $I \subseteq A \times B$. Denote by x_{ij} and u_{ij} generic elements of A_{ij}, and by y_{ij} and v_{ij} elements of B_{ij}. We turn I into a *signed space* by defining, for every $(x_{ik}, y_{kj}) \in I$,

$$\operatorname{sign}(x_{ik}, y_{kj}) = \operatorname{sign}(x_{ik}) \operatorname{sign}(y_{kj}).$$

Let $\mathbf{A} = ||a_{ij}||$, and $\mathbf{B} = ||b_{ij}||$, and suppose that $\mathbf{AB} = \mathbf{I}$. In many instances (for example, in the combinatorics of the transition matrices for symmetric function bases), one can provide a combinatorial proof of this fact (viz., $\mathbf{AB} = \mathbf{I}$) by means of a *sign-reversing involution* on I. That is, we can construct a permutation α on I satisfying

(i) $\alpha : I_{ij} \to I_{ij}$;

(ii) if $\alpha(x_{ik}, y_{kj}) = (u_{il}, v_{lj}) \neq (x_{ik}, y_{kj})$, then $\operatorname{sign}(x_{ik}, y_{kj}) \neq \operatorname{sign}(u_{il}, v_{lj})$ (that is, α is sign-reversing outside its fixed point set);

(iii) α has no fixed points in I_{ij} for $i \neq j$, and a unique fixed point (with positive sign) in I_{ii} for each i;

α is an involution, that is, α^2 is the identity.

Then it is easy to see, by using α, that

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Since $\mathbf{AB} = \mathbf{I}$ if and only if $\mathbf{BA} = \mathbf{I}$, it is natural to ask if $\mathbf{BA} = \mathbf{I}$ admits a combinatorial proof via a sign-reversing involution β defined on

$$J = \bigcup_{i,j=1}^{n} J_{ij},$$

where

$$J_{ij} = \bigcup_{k=1}^{n} B_{ik} \times A_{kj},$$

where β is constructed directly from α. We would also like the construction to be *natural*, in the sense that the map $\alpha \to \beta$ itself should be an involution.

Problem: Give such a construction of β from α.

Problem 9 (BCC12.9) Subgraphs of the n-cube. Proposed by Paul Erdős.

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Let Q_n denote the n-cube, a graph with $n2^{n-1}$ edges.

(a) Show that a subgraph of Q_n with at least $(\frac{1}{2} + \epsilon)n2^{n-1}$ edges contains a C_4.

(b) Show that a subgraph of Q_n with at least $\epsilon n2^{n-1}$ edges contains a C_6.

Of course, the assertion is to be shown for all sufficiently large n, for any given $\epsilon > 0$. Each of these problems is worth \$100.

Problem 10 (BCC12.10) Sum-free sets containing a small even number. Proposed by Paul Erdős.

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Let S be a sum-free subset of $\{1, \ldots, n\}$ (that is, for all $x, y, z \in S$, $x + y \neq z$). Clearly $|S| \leq \lceil \frac{1}{2}n \rceil$. Now suppose that S contains an even number less than $\frac{1}{2}n - f(n)$. Is it true that $|S| \leq \frac{1}{2}n - \epsilon f(n)$?

The hypothesis is clearly necessary: the set of odd numbers, and the set of numbers greater than $\frac{1}{2}n$, are both sum-free. It is known that, if $2 \in S$, then $|S| \leq (\frac{2}{5} + o(1))n$.

Problem 11 (BCC12.11) Forbidding $x + y = 4z$. Proposed by Paul Erdős.

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Let S be a set of positive integers containing no solution to $x + y = 4z$. Does S have lower density less than $\frac{1}{2}$?

Note that the set of integers congruent to 1 or 4 modulo 5 satisfies the constraint and has density $\frac{2}{5}$.

Problem 12 (BCC12.12) Sets with all subset-sums distinct. Proposed by Paul Erdős.

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Let S be a subset of $\{1, \ldots, n\}$ with all sums of subsets distinct. Is it true that

$$|S| < \frac{\log n}{\log 2} + c?$$

It is known that $|S| < \frac{\log n}{\log 2} + \frac{\log \log n}{2 \log 2}$, and that there is such a set with $n = 2^m$ and $|S| = m + 2$. This problem is worth \$500.

Problem 13 (BCC12.13) Some distance-regular graphs. Proposed by Tony Gardiner.

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Is there a distance-regular graph with diameter 3 having intersection array

$$\begin{pmatrix} * & 1 & c & m \\ 0 & 0 & a & 0 \\ m & m-1 & b & * \end{pmatrix}$$

with $b \neq 1$, $a \neq 0$?

The ith column of the intersection array gives the number of vertices adjacent to y and at distance $i-1$, i, $i+1$ respectively from x, where x and y are at distance i these numbers being (by definition) constant. The smallest such feasible graph has valency 126.

Problem 14 (BCC12.14) Determining a graph from its cycle space. Proposed by Oliver Pretzel.

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Given a cycle basis of a graph G, one can view the cycles of G as elements of a free abelian group (or, if orientation is disregarded, of a vector space over $\mathbb{Z}/2$). Many functions on the cycles are linear on this algebraic structure, but the length function (the number of edges in the cycle) is not. Informally, the question is: "how closely does the length function determine G?"

More formally, suppose that two graphs G and H are given with isomorphic cycle spaces. Suppose further that G is known and that, for suitable cycle bases, the length functions on the cycle spaces of G and H are the same. What can be said about H?

For example, if G is a tree, then H is a forest. If G is a cycle with two chords and H is 2-connected, then H is a cycle of the same length with two chords, but the chords can be moved slightly. So H need not be isomorphic to G.

Conjecture: If G and H are 3-connected and have the same length function for cycles, then G and H are isomorphic

Problem 15 (BCC12.15) A double Youden rectangle. Proposed by Donald A. Preece.

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Construct a 5×11 double Youden rectangle.

For the definition, see Bailey [1]. Briefly: what is required is a 5×11 rectangle, each cell containing a symbol from $\{a_1, \ldots, a_{11}\}$ and a symbol from $\{b_1, \ldots, b_5\}$, such that

(i) the a's form a Latin rectangle, and the sets of a's appearing in the columns are the blocks of a symmetric $(11, 5, 2)$ BIBD;

(ii) each b appears once in each column and once paired with each a;

(iii) For $i \neq j$, the numbers of occurrences of b_i and b_j in any row differ by at most 1.

Of course, this definition is much less "symmetrical" than that in Bailey [1].

Problem 16 (BCC12.16) Monotone directed paths in tournaments. Proposed by V. Linek, B. Sands, N. Sauer and R. E. Woodrow.

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Problem 1: If the edges of a tournament are coloured with three colours, is there a set of three vertices such that there is a monochromatic directed path from any other vertex to one of these three?

It is known that, if only two colours are used, then a single vertex will suffice. For three colours, three vertices are necessary but it is not known whether any finite number is sufficient. See [6].

Problem 2: Let P be a poset, and colour the edges of a tournament with the elements of P. Call a directed path $x_1, x_2, \ldots, x_k$ *monotone* if

$$\text{colour}(x_i x_{i+1}) \leq \text{colour}(x_{i+1} x_{i+2})$$

for all i. If P is the poset $\{\overset{\bullet}{\underset{\bullet}{|}}\quad\bullet\}$, is there a set of three vertices of T such that there is a monotone directed path from any other vertex of T to one of these three?

Again it is known that two vertices will not always do, but no fixed number is known to suffice. Problem 1 is just Problem 2 with P a 3-element antichain. Linek and Sands have found all posets P for which a *single* vertex suffices; they are exactly those containing neither $\{\bullet\quad\bullet\quad\bullet\}$ nor $\{\overset{\bullet}{\underset{\bullet}{|}}\quad\bullet\}$.

Problem 3: Problem 2 can be generalised by replacing the poset P by a directed graph D, and replacing "monotone" paths by paths $x_1, \ldots, x_k$ such that

$$(\text{colour}(x_i x_{i+1}), \text{colour}(x_{i+1} x_{i+2})) \text{ is an edge of } D$$

for all i. For example, if $D = C_5$, the undirected 5-cycle (with each edge directed both ways), is there always a vertex in any tournament reachable from all others by a D-path?

Problem 17 (BCC12.17) Independence numbers in non-Hamiltonian graphs. Proposed by Douglas R. Woodall.

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Let G be a k-connected graph with vertex independence number $\alpha(G)$.

Conjecture 1: If G is not Hamiltonian, C is a longest circuit in G, and $k \geq 2$, then

$$\alpha(G - C) \leq \alpha(G) - k.$$

Conjecture 2: If G is not traceable, P is a longest path in G, and $k \geq 1$, then

$$\alpha(G - P) \leq \alpha(G) - k - 1.$$

Here, $G - C$ means the graph obtained by removing the *vertices* of C from G.

If true, both conjectures are best possible: let G be the union of a large number of complete graphs, all having exactly k vertices in common. Conjecture 2 is obvious if $k = 1$ (since the end vertices of P are not adjacent to each other or to anything outside P), and has been proved for $k = 2$, but is open for $k \geq 3$, as is Conjecture 1 in all cases.

Problem 18 (BCC12.18) Zara graphs. Proposed by François Zara.

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A *Zara graph* is a graph with the properties that, for some natural numbers r and t,

(i) each maximal clique is of cardinality r;

(ii) if C is a maximal clique and x a vertex not in C, then x is adjacent to exactly t vertices in C.

Problems:

1. Find new Zara graphs.
2. A Zara graph with $t = 2$ is a special kind of "extended generalised quadrangle" (see [5]). Can these graphs be classified?

For further background, see [2, 9, 10].

Editor's note: At the Second Franco-Chinese Conference on Counting and Coding, held at ENST, Paris, 6–8 September 1989, Tayuan Huang (Department of Applied Mathematics, National Chiao-Tung University, Hsin Chu 30050, Taiwan, ROC; `thuang@twnctu01.bitnet`) proposed a solution to Problem 1, involving a family of Zara graphs with $r = q^3$, $t = q^2$ for any prime power q, constructed from alternating bilinear forms in 4 variables over $GF(q)$. (Join two forms if their difference is singular.) Subsequently, it turned out that these are isomorphic to some of the known examples.

Of course, by definition, Problem 1 remains open!

Final note. Participants at the problem session will recall some discussion of a problem posed by Oliver Pretzel. Subsequently, a message reached me by an indirect route from Noga Alon, pointing out that the problem had been solved "decades ago" by Erdős and Selfridge. The proof is elegant, and participants may be interested in seeing it.

The problem was as follows. Let A and B be distinct finite sets of positive integers for which the multisets

$$A_2 = \{a + a' : a, a' \in A, a \neq a'\}, \quad B_2 = \{b + b' : b, b' \in B, b \neq b'\}$$

coincide. Is it true that $|A| = |B|$ is a power of 2?

Solution. Put $f(x) = \sum_{a\in A} x^a$ and $g(x) = \sum_{b\in B} x^b$. Then

$$\begin{aligned} f(x)^2 - f(x^2) &= 2\sum_{r\in A_2} x^r, \\ g(x)^2 - g(x^2) &= 2\sum_{s\in B_2} x^s. \end{aligned}$$

Hence $f(x)^2 - f(x^2) = g(x)^2 - g(x^2)$, that is, $(f-g)(x)(f+g)(x) = (f-g)(x^2)$. Suppose that $(f-g)(x) = (1-x)^k P(x)$ where $P(1) \neq 0$. Then

$$(f = g)(x) + \frac{(f-g)(x^2)}{(f-g)(x)} = (1+x)^k \frac{P(x^2)}{P(x)}.$$

Put $x = 1$ to get

$$2|A| = (f+g)(1) = 2^k \frac{P(1)}{P(1)} = 2^k,$$

so the answer is affirmative.

References

[1] R. A. Bailey, Designs: mappings between structured sets, pp. 22–51 in *Surveys in Combinatorics, 1989* (ed. J. Siemons), Cambridge Univ. Press, Cambridge, 1989.

[2] A. Blokhuis and H. Wilbrink, Characterization theorems for Zara graphs, *Europ. J. Combinatorics* **10** (1989), 57–68.

[3] P. J. Cameron and P. Erdős, On the number of sets of integers with various properties, pp. 61–79 in *Number Theory* (ed. R. A. Mollin), Walter de Gruyter, Berlin, 1990.

[4] P. J. Cameron, P. Frankl and W. M. Kantor, Intersecting families of finite sets and fixed-point-free 2-elements, *Europ. J. Combinatorics* **10** (1989), 149–160.

[5] P. J. Cameron, D. R. Hughes and A. Pasini, Extended generalized quadrangles, *Geometriae Dedicata* **35** (1990), 193–228.

[6] B. Sands, N. Sauer, and R. E. Woodrow, On monochromatic paths in edge-coloured digraphs, *J. Combinatorial Theory* (B) **33** (1982), 271–275.

[7] V. Linek and B. Sands, A note on paths in edge-coloured tournaments, *Ars Combinatoria* **44** (1996), 225-228.

[8] Meyerowitz, Aaron Maximal intersecting families. (English) Eur. J. Comb. 16, No.5, 491-501 (1995).

[9] F. Zara, Graphes lies aux espaces polaires, Eur. J. Comb. 5, 255-290 (1984). *Europ. J. Combinatorics* **5** (1984), 255–290.

[10] F. Zara, Graphes lies aux espaces polaires, II, *Europ. J. Combinatorics* **8** (1987), 335–340.