Solutions to odd-numbered exercises
Peter J. Cameron, *Introduction to Algebra*, Chapter 5

5.1 All of the left and right module axioms are easily demonstrated by using the associative, distributive, zero, and additive inverse laws for matrices, while the extra bimodule axiom comes from the associative law. (Note that this bimodule is unital for both left and right actions.)

5.3 (a) Suppose that M is generated as R-module by the element m. Define a map $\theta : R \to M$ by

$$r\theta = mr$$

for all $r \in R$. We claim that θ is an R-module homomorphism. This involves checking that

$$\begin{aligned} m(r_1 + r_2) &= mr_1 + mr_2, \\ m(r_1 r_2) = (mr_1)r_2, \end{aligned}$$

which are just module axioms (MM2) and (MM3). [In the case of the second axiom, we have to show that $(r_1 r_2)\theta = (r_1\theta)r_2$. We are thinking of r' as an element of the free R-module R, and r_2 as an element of the ring R acting on this module.] The kernel of θ is $\{r \in R : mr = 0\}$. Clearly any element $r \in \mathrm{Ann}(M)$ satisfies this condition. Conversely, since m generates M, every element of M has the form mr' for some $r' \in R$; then $mr = 0$ implies $(mr')r = (mr)r' = 0$, so $r \in \mathrm{Ann}(M)$. [Remember that our rings are now commutative with identity!] So the $\mathrm{Ker}(\theta) = \mathrm{Ann}(M)$. Also, since m generates M, we have $\mathrm{Im}(\theta) = M$. So $M \cong R/Ann(M)$ (as R-modules).

5.5 Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the characteristic polynomial of A is $x^2 - (a+d)x + (ad - bc)$; and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

after some calculation.

The 3×3 case is left to you!

5.7 We know that F^n, as $F[x]$-module (as in Example 6 on page 185) is isomorphic to the direct sum of the modules $F(x)/\langle f_i(x)\rangle$, where $f_i(x)$ are the diagonal entries in the Smith normal form.

None of the polynomials $f_i(x)$ can be zero; for $F[x]/\langle 0\rangle = F[x]$ is an infinite-dimensional vector space over F, whereas F^n is finite-dimensional. So $f_1(x), \dots, f_n(x)$ are the invariant factors.

If $f_i(x) = 1$, then the corresponding summand is $F[x]/\langle 1\rangle = \{0\}$, and can be removed from the direct sum without changing it. So we are left with just the non-constant polynomials $f_i(x)$.

5.9 (a) The set X in question is just $\{v \in V : vA = \lambda v\}$, since the eigenvectors are precisely the non-zero vectors in this set.

But, if $vA = \lambda v$ and $wA = \lambda w$, then

$$\begin{aligned} (v+w)A = vA + wA &= \lambda v + \lambda w = \lambda(v+w), \\ (cv)A = c(vA) &= c(\lambda v) = \lambda(cv); \end{aligned}$$

So X passes the Subspace Test.

(b) Suppose that we have a linear dependence relation

$$c_1 v_1 + \cdots + c_k v_k = 0,$$

where $c_1, \ldots, c_k$ are not all zero. We may assume that the number of non-zero elements among $c_1, \ldots, c_k$ is as small as possible for any linear dependence. The number of non-zero elements must be at least two. For suppose that all except c_i were zero. Then we would have, say, $c_i v_i = 0$, contradicting the fact that $c_i \neq 0$ and $v_i \neq 0$ (as v_i is an eigenvector).

Without loss of generality, we can suppose that c_1 and c_2 are non-zero. Multiplying the displayed equation by A, using the fact that $v_i A = \lambda_i v_i$, we obtain

$$\lambda_1 c_1 v_1 + \lambda_2 c_2 v_2 + \cdots + \lambda_k c_k v_k = 0.$$

Multiplying the displayed equation by λ_1, we obtain

$$\lambda_1 c_1 v_1 + \lambda_1 c_2 v_2 + \cdots + \lambda_1 c_k v_k = 0.$$

Subtracting these two equations gives

$$(\lambda_2 - \lambda_1) c_2 v_2 + \cdots + (\lambda_k - \lambda_1) c_k v_k = 0.$$

But this last relation is a linear dependence relation on $v_1, \ldots, v_k$ (since the coefficient of v_2 is $(\lambda_2 - \lambda_1) c_2 \neq 0$), with one fewer non-zero term than the equation with which we began. This contradicts the fact that we started with as few non-zero terms as possible.

5.11 (a) implies (b): If λ is an eigenvalue of A, then there is a non-zero vector such that $vA = \lambda v$. Then $v(A - \lambda I) = O$, so $A - \lambda I$ is not invertible; so its determinant is zero. But this determinant is just $c(\lambda)$.

(b) implies (c): Suppose that $c(\lambda) = 0$, but $m(\lambda) \neq 0$. Then $\det(A - \lambda I) = 0$, so $A - \lambda I$ is not invertible; thus there exists a vector $v \neq 0$ with $vA = \lambda v$. But then an easy argument shows that $vm(A) = m(\lambda)v \neq 0$, contradicting the fact that $m(A) = O$.

(c) implies (a): If $m(\lambda) = 0$, then $c(\lambda) = 0$ by the Cayley–Hamilton Theorem 5.21. Then just as in the case (b) implies (c), we see that there exists an eigenvector with eigenvalue λ.

Remark: In this proof, you see that we are covering some of the ground more than once. The direction of proof given in the book seems to be the most natural!

5.13 Let $d_1, \dots, d_n$ be the diagonal elements in the Smith normal form of A. By Theorem 5.9,

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z}.$$

We see immediately that, if some d_i is zero, then G is infinite, while if all d_i are finite, then $|G|$ is just their product.

Now, when we apply an elementary row or column operation to a matrix, the effect on the determinant (if any) is to multiply it by -1 or a unit. In $\mathbb{Z}$, the only units are $+1$ and -1. So the determinant of the Smith normal form of A (which is $d_1 d_2 \cdots d_n$) is equal, up to sign, to $\det(A)$, and we are done.