Solutions to odd-numbered exercises
Peter J. Cameron, *Introduction to Algebra*, Chapter 4

4.1 Using the Second Subspace Test, we have to decide whether it is true that, if f_1 and f_2 are bounded, or continuous, or differentiable, or satisfy $f(0) = f(1)$, then the same is true for $c_1 f_1 + c_2 f_2$ for all real numbers c_1 and c_2.

(a) To say that f_1 and f_2 are bounded means that there exist M_1 and M_2 such that $|f_1(x)| \leq M_1$ and $|f_2(x)| \leq M_2$ for all $x \in [0,1]$. Then

$$c_1 f_1(x) + c_2 f_2(x)| \leq |c_1| \cdot |f_1(x)| + |c_2| \cdot |f_2(x)| \leq |c_1|M_1 + |c_2|M_2,$$

using properties of the modulus function:

$$|a+b| \leq |a| + |b|, qquad |ab| = |a| \cdot |b|.$$

So $c_1 f_1 + c_2 f_2$ is bounded.

The affirmative answer in (b) and (c) follows from elementary calculus.

For (d), if $f_1(0) = f_1(1)$ and $f_2(0) = f_2(1)$, then

$$c_1 f_1(0) + c_2 f_2(0) = c_1 f_1(1) + c_2 f_2(1),$$

so again we have a subspace.

4.3 (a) The set $\{0\}$ is a subspace since if $v_1 = v_2 = 0$ then $c_1 v_1 + c_2 v_2 = 0$. (This is true in any vector space.)

A line L through the origin consists of all the vectors of the form av for some fixed non-zero vector v and arbitrary real numbers a. Now

$$c_1(a_1 v) + c_2(a_2 v) = (c_1 a_1 + c_2 a_2)v \in L,$$

so L is a subspace.

It is clear that the whole set V is a subspace.

Now let U be an arbitrary subspace. Then U must be non-empty, and so must contain the zero vector. (If $u \in U$, then $0 = u - u \in U$.) If there are no other vectors in U, then it is the set $\{0\}$.

If v is a non-zero vector in U, then every scalar multiple av is contained in U. If there are no other vectors in U, then it is a line through the origin.

Suppose that U contains a vector w which is not a scalar multiple of v. Then any vector in the plane is a linear combination of v and w [WHY?], and so $U = V$.

4.5 A vector in $U \cap V$ can be written in either of the two forms

$$a(1,2,0,-1) + b(2,1,1,3) + c(1,-1,1,2) = d(3,2,0,2) + e(2,2,0,1).$$

This gives us four equations for the five numbers a, b, c, d, e, whose solutions will give us the intersection. The equations are

$$\begin{aligned} a + 2b + c &= 3d + 2e, \\ 2a + b - c &= 2d + 2e, \\ b + c &= 0, \\ -a + 3b + 2c &= 2d + e. \end{aligned}$$

Using the third equation to eliminate c gives

$$\begin{aligned} a+b &= 3d+2e, \\ 2a+2b &= 2d+2e, \\ -a+b &= 2d+e. \end{aligned}$$

The first two equations then tell us that $6d+4e = 2d+2e$, so $e = -2d$; so we have

$$\begin{aligned} a+b &= -d, \\ a-b &= 0. \end{aligned}$$

So $a = b$, and everything can be expressed in terms of a:

$$b = a, \qquad c = -a, \qquad d = -2a, \qquad e = 4a.$$

Then the typical vector in the intersection has the form

$$a(1,2,0,-1)+a(2,1,1,3)-a(1,-1,1,2) = -2a(3,2,0,2)+4a(2,2,0,1) = a(2,4,0,0).$$

So $(2,4,0,0)$ is a basis for $U \cap W$.

Remark: We have gone about this in a rather unsystematic way. Can you formulate some general rules for such a calculation?

4.7 The First Isomorphism Theorem 4.14 asserts:
Let $\theta : V \to W$ be a linear transformation of vector spaces. Then

(a) $\mathrm{Im}(\theta)$ *is a subspace of W;*

(b) $\mathrm{Ker}(\theta)$ *is a subspace of V;*

(c) $V/\mathrm{Ker}(\theta) \cong \mathrm{Im}(T)$.

Note that, unlike for groups and rings (where kernels of homomorphisms are subgroups or subrings with some additional property, viz., normal subgroups or ideals), in this case no extra property is required. Also, the proof is much simpler. A vector space of dimension n over a field F is isomorphic to F^n. So to prove the isomorphism in (c), all we have to do is to establish that the two vector spaces have the same dimension; this is just the Rank and Nullity Theorem 4.15.

However, you are encouraged to write a more "structural" proof following the proofs of the theorem for rings and groups.

The Second Isomorphism Theorem states:

Let W be a subspace of a vector space V. Then there is a one-to-one correspondence between the set of subspaces of V containing W and the set of subspaces of V/W.

Suppose that U is a subspace of V containing W, and consider U/W (the set of cosets of W contained in U: note that any coset of W is either contained in U or disjoint from U). Take two cosets $W+u_1$ and $W+u_2$ in U/W. We have

$$c_1(W+u_1)+c_2(W+u_2) = W+(c_1u_1+c_2u_2) \in U/W,$$

since U is a subspace; so U/W is a subspace. The converse is similar.

The Third Isomorphism Theorem states:

Let U and W be two subspaces of V. Then $U \cap W$ and $U + W$ are subspaces of V; and $(U+W)/W \cong U/(U \cap W)$.

The first part is straightforward, and the dimension argument proving the second is Theorem 4.11. (Again you are encouraged to write out a more structural proof.)

4.9 The greatest common divisor of the elements of the matrix is 1. The gcd of the 2×2 subdeterminants is 30, while the determinant of the matrix is 1800. So the Smith normal form must be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 60 \end{pmatrix}.$$

The exercise involves finding the right row and column operations to give this result; I leave this to you!