Queen Mary University of London

MTHM024/MTH714U **Group Theory**

Notes 3 **Autumn 2011**

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3 Proof of the Jordan–Hölder Theorem

Recall that we are proving that any two composition series for a group G have the same length and give rise to the same list of composition factors.

The proof is by induction on the order of G. We suppose the theorem true for groups smaller than G. Let

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_r = \{1\}$$

and

$$G = H_0 \rhd H_1 \rhd H_2 \rhd \cdots \rhd H_s = \{1\}$$

be two composition series for G.

Case 1: $G_1 = H_1$. Then the parts of the series below this term are composition series for G_1 and so have the same length and composition factors. Adding in the composition factor G/G_1 gives the result for G.

Case 2: $G_1 \neq H_1$. Let $K_2 = G_1 \cap H_1$, a normal subgroup of G, and take a composition series

$$K_2 \rhd K_3 \rhd \cdots \rhd K_t = \{1\}$$

for K_2.

We claim that $G_1/K_2 \cong G/H_1$ and $H_1/K_2 \cong G/G_1$. If we can prove this, then the two composition series

$$G_1 \rhd G_2 \rhd \cdots \rhd \{1\}$$

and

$$G_1 \rhd K_2 \rhd K_3 \rhd \cdots \rhd \{1\}$$

for G_1 have the same length and composition factors; the composition factors of G using the first series are these together with G/G_1. A similar remark holds for H_1. So each of the given composition series for G has the composition factors in the series for K_2 together with G/G_1 and G/H_1, and the theorem is proved. So it only remains to establish the claim.

Now G_1H_1 is a normal subgroup of G properly containing G_1; so $G_1H_1 = G$. Thus, by the Third Isomorphism Theorem,

$$G/G_1 = G_1H_1/G_1 \cong H_1/G_1 \cap H_1 = H_1/K_2,$$

and similarly $G/H_1 \cong G_1/K_2$. Thus the claim is proved.

The following picture might make this proof clearer. The last part of the proof shows that the quotient groups corresponding to opposite sides of the parallelogram at the top of the figure are isomorphic.