Queen Mary
University of London

C50 Enumerative & Asymptotic Combinatorics

Solutions to Exercises 8 — Spring 2003

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1 For $S(n,k)$, we can use the formula

$$S(n,k) = \frac{1}{k!}\sum_{i=0} k-1(-1)^i \binom{k}{i} (k-i)^n$$

(count the number of surjective functions from $\{1,\ldots,n\}$ to $\{1,\ldots,k\}$ by inclusion-exclusion, and divide by $k!$). Clearly the term k^n is exponentially larger than any other term in the sum (the number of terms and the coefficients depend on k, which is fixed, and $(k-i)^n = o(k^n)$ for $i > 0$). So $S(n,k) \sim n^k/k!$.

A partition of $\{1,\ldots,n\}$ into $n-k$ parts, with k fixed (say $k < n/2$) has the property that all but at most k of the parts consist of singletons. First, count partitions in which all parts have size 1 or 2: there are k parts of size 2, whose union has $2k$ points. These points can be chosen in $\binom{n}{2k} \sim n^{2k}/(2k)!$ ways, and the set of $2k$ points can be partitioned into pairs in $1.3.\cdots(2k-1) = (2k)!/2^k k!$ ways. Thus the number of partitions of this kind is asymptotically $n^{2k}/2^k k!$. The remaining partitions have fewer than $2k$ points in parts of size bigger than 1; the number of them is at most $\binom{n}{2k-1}(2k-1)!$ which is of smaller order than what we already found.

The argument gives the same estimate for $|s(n,n-k)|$: partitions with parts of size 1 or 2 correspond bijectively with permutations with cycle lengths 1 or 2, and we overestimated the others in a way that applies to permutations as well.

The asymptotic estimate for $s(n,k)$ is a bit harder. Try the case $k = 2$: we have

$$|s(n,2)| = \sum_{m=1}^{\lfloor n/2 \rfloor} \binom{n}{m} (m-1)!(n-m-1)! = n! \sum \frac{1}{m(n-m)}.$$

Can you evaluate this sum?

2 The recurrence relation for the Bernoulli numbers shows that $(n+1)B_n$ is an integer linear combination of Bernoulli numbers with smaller index. By the induction hypothesis, $n!$ times the right-hand side is an integer. So $(n+1)!B_n$ is an integer.

3 Equation (b) for the Bernoulli polynomials shows that

$$\frac{1}{k+1}(B_k(t+1) - B_k(t)) = t^k.$$

Summing these equations for $t = 1, dots, n$ and using $B_k(1) = B_k(0)$, gives the result.

4 With $f(x) = 1/x$ we have $f^{(k)}(x) = (-1)^k x!\, x^{-(k+1)}$. Moreover, $\int_1^n f(t)\,\mathrm{d}t = \log n$. Substitution gives the required series, apart from an unknown constant which you are not expected to evaluate.

5 Let t_n be the number of fixed-point-free involutions on n points. We found earlier that

$$t_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 \cdot 3 \cdot 5 \cdots (n-1) & \text{if } n \text{ is even,} \end{cases}$$

so that the exponential generating function of (t_n) is $\exp(x^2/2)$. Now

$$s_n = \sum_{k=0}^{n} \binom{n}{k} t_{n-2k},$$

so the e.g.f. of (s_n) is the product of those for (1) and (t_n), that is, $\exp(x) \cdot \exp(x^2/2) = \exp(x + x^2/2)$.

The application of Hayman's Theorem is a bit delicate: there is a full discussion in Odlyzko's paper.