C50 Enumerative & Asymptotic Combinatorics

Solutions to Exercises 7 — Spring 2003

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1 Using the Prüfer codes, we see that a tree has this property if and only if the symbol i occurs $a_i - 1$ times in its code. So to count the trees, we count the $n-2$-tuples containint $a_i - 1$ occurrences of i for $i = 1, \ldots, n$; this is the *multinomial coefficient*

$$\binom{n}{a_1-1, a_2-1, \ldots, a_n-1} = \frac{n!}{(a_1-1)!\,(a_2-1)!\cdots(a_n-1)!}.$$

2 There is one circular structure on n points, with cycle index

$$\frac{1}{n}\sum_{m|n}\phi(m)s_m^{n/m}$$

for $n > 1$. So

$$\begin{aligned}
Z(\mathcal{C}) &= 1 + \sum_{n\geq 1}\frac{1}{n}\sum_{m|n}\phi(m)s_m^{n/m} \\
&= 1 + \sum_{m\geq 1}\frac{\phi(m)}{m}\sum_{k\geq 1}\frac{1}{k}s_m^k \\
&= 1 - \sum_{m\geq 1}\frac{\phi(m)}{m}\log(1-s_m).
\end{aligned}$$

(We put $n = mk$ and sum over m and k in the second line.)

Using the fact that $Z(\mathcal{S}) = \exp(\sum_{i\geq 1}\frac{s_i}{i})$, we obtain

$$\begin{aligned}
Z(\mathcal{P}) &= \exp\Big(\sum_{i\geq 1}\frac{-\sum_{j\geq 1}(\phi(j)/j)\log(1-s_{ij})}{i}\Big) \\
&= \prod_{i\geq 1}\prod_{j\geq 1}(1-s_{ij})^{\phi(j)/ij} \\
&= \prod_{n\geq 1}(1-s_n)^{-1}.
\end{aligned}$$

In the last line we put $n = ij$ and note that $\sum_{j|n}\phi(j) = n$.

For a direct proof of this result we note that the automorphism group of a permutation is its centraliser in the symmetric group, so that

$$\mathcal{Z}(\mathcal{P}) = \sum_{n\geq 0}\sum_{g\in S_n} \frac{|C(g)|}{n!}\frac{1}{|C(g)|}\sum_{h\in C(g)} z(h),$$

writing $C(g)$ for the centraliser of g in S_n. (The coefficient $|C(g)|/n!$ is because isomorphism classes are the conjugacy classes in S_n, not individual elements.) Now we can reverse the order of summation: the coefficient of $z(h)$ will then be the sum of $1/n!$ over all $g \in C(h)$, which is $|C(h)|/n!$, or the reciprocal of the size of the conjugacy class of h. So the final sum is just over conjugacy class representatives. Thus each monomial has coefficient 1, and we have

$$\begin{aligned}
\mathcal{Z}(\mathcal{P}) &= \sum_{c_1,c_2,\ldots}\prod_i s_i^{c_i} \\
&= \prod_i \sum_c s_i^c \\
&= \prod_i (1-s_i)^{-1}
\end{aligned}$$

as required.

3 Since $c_n = 1$ for all n, the ordinary generating function for unlabelled structures in $\mathcal{C}$ is

$$\sum_{n\geq 0} x^n = 1 + \frac{x}{1-x}.$$

We must get the same result by substituting $s_n \leftarrow x^n$ in the cycle index, giving

$$1 - \sum_{m\geq 1}\frac{\phi(m)}{m}\log(1-x^m).$$

Equating, cancelling the 1, and taking the exponential, gives

$$\prod_{m\geq 1}(1-x^m)^{-\phi(m)/m} = \exp\left(\frac{x}{1-x}\right).$$

4 The required generating function is

$$\begin{aligned}
\mathcal{Z}(\mathcal{S})(s_n \leftarrow g(x^n)-1) &= \exp\sum_{m\geq 1}\frac{g(x^m)-1}{m} \\
&= \exp\sum_{m\geq 1}\sum_{i\geq 1}\frac{g_i x^{im}}{m} \\
&= \exp\sum_{i\geq 1} g_i \sum_{m\geq 1}\frac{x^{im}}{m} \\
&= \exp\sum_{i\geq 1} -g_i\log(1-x^i) \\
&= \prod_{i\geq 1}(1-x^i)^{-g_i},
\end{aligned}$$

as required.

If $\mathcal{G} = \mathcal{S}$ the generating function is $\prod_{i\geq 1}(1-x^i)^{-1}$, which we recpgnise as the generating function $\sum_{n\geq 0} p(n)x^n$ for the partition numbers $p(n)$. Now an $\mathcal{S}[\mathcal{S}]$-object is a set with a partition, with no structure on either the individual parts or the set of parts; so the number of unlabelled structures is just $p(n)$, as it should be.

5 No solution given here.

6 If we have a rooted tree with root $n+1$, let $i_1,\ldots,i_k$ be the neighbours of the root. Deleting the root gives k rooted trees with roots $i_1,\ldots,i_k$. Conversely, given collection of rooted trees on n vertices, we recover a tree on $n+1$ vertices by adding a new root $n+1$ joined to all the roots in the given trees.

Thus, if $R(x)$ is the e.g.f. for labelled rooted trees, the number of rooted trees on $n+1$ vertices with root $n+1$ is $R_{n+1}/(n+1)$, and so the probability that a rooted forest on n vertices is connected is

$$\begin{aligned} \frac{(n+1)R_n}{R_{n+1}} &= \frac{(n+1)n^{n-1}}{(n+1)^n} \\ &= \left(1-\frac{1}{n+1}\right)^{n-1} \\ &\to \frac{1}{\mathrm{e}} \quad \text{as } n\to\infty. \end{aligned}$$

7 A member of the species $\mathcal{U}$ consists of a set with a distinguished subset; its automorphism group consists of all permutations which fix the distinguished subset. Thus $\mathcal{U}$ is equivalent to $\mathcal{S}\times\mathcal{S}$. Its cycle index and generating functions are thus the squares of the corresponding series for $\mathcal{S}$.