Queen Mary
University of London

C50 Enumerative & Asymptotic Combinatorics

Solutions to Exercises 6 — Spring 2003

The first three questions are not very interesting.

4 $k!\,S(n,k)$ is equal to the number of surjections from an n-set to a k-set; for, given a partition of $\{1,\ldots,n\}$ into k parts, we can number the parts $1,\ldots,k$ in $k!$ ways, and each numbering corresponds to the surjection mapping each element to the number of the part containing it.

There are $k-j)^n$ functions whose range excludes a given set of j points. Now inclusion-exclusion gives the result.

5 let $\lambda(x,y)$ be the alternating sum on the right of the proposed equality. Clearly $\lambda(x,x)=1$ since there is a unique chain of length 0. We have to prove that

$$\lambda(x,y) = -\sum_{x\le z<y} \lambda(x,z),$$

from which the equation $\lambda=\mu$ follows by induction. Now any chain of length c with second-last element z contributes $(-1)^c$ to $\lambda(x,y)$ and also contributes $(-1)^{c-1}$ to $\lambda(x,z)$. So the equation is true.

6 By Möbius inversion, this is equivalent to proving that

$$d(n) = \sum_{m|n} 1,$$

which is obvious.

7 (a) Note that we are asked to show that $\mu(E,P)$ is equal to the number of permutations whose cycle decomposition is the partition P, multiplied by the sign of such a permutation.

The proof of the formula is by induction on n. The start of the induction is trivial.

If P has more than one part, then the interval $[E,P]$ is isomorphic to the product of intervals of the form $[E,X_i]$ for sets X_i with $|X_i| = a_i$, and the result follows from the product formula and the induction hypothesis.

If there is only a single part, use the fact that $\sum_{g\in S_n} \mathrm{sign}(g) = 0$ for $n>1$. Now by the induction hypothesis, the sum of $\mu(E,Q)$ over all $Q<P$ is equal to the sum of the signs of all permutations having more than one cycle; this is equal to the negative of the sum of signs of

permutations with a single cycle, giving $\mu(E,P) = (-1)^{n-1}(n-1)!$, as required. (There are $(n-1)!$ such permutations each with sign $(-1)^{n-1}$.)

(b) Consider the interval $[P.Q]$. Each part of Q splits into a certain number a_i of parts of P, and the interval is isomorphic to the product of partition lattices on a_i points for all relevant i. Thus we obtain the same formula as in (a) for $\mu(P,Q)$ but with a_i being the number of parts of P in a given part of Q, rather than the total number of points in such a part.

8 The mth power of g consists of n/d cycles each of length d, where $d = \gcd(m,n)$. For each d dividing n, the number of m with $\gcd(m,n) = d$ is $\phi(n/d)$: these are all numbers of the form dx, where $\gcd(x,n/d) = 1$. So the cycle index is as claimed.

9 Immediate from the Cycle Index Theorem.