Queen Mary University of London

C50 Enumerative & Asymptotic Combinatorics

Solutions to Exercises 5 — Spring 2003

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1 The average number of fixed points of an element of G is equal to the number of orbits of G, which is 1. Since $n > 1$, there is an element (the identity) which fixes more than one point. So there must be an element which fixes less than one point.

The argument for the next part can be phrased in many ways; here it is in terms of prpobability theory. Let g denote a random element of G (chosen from the uniform distribution) and f the random variable for which $f(g)$ is the number of fixed points of g. Then we have $E(f) = 1$ (by the orbit-counting lemma). Also, $E(f^2) \geq 2$, since $f(g)^2$ is the number of fixed points of g on X^2, and the number of orbits here is at least two (since pairs (x,x) and (x,y) (for $x \neq y$) lie in different orbits. Hence

$$E((f-1)(f-n)) = E(f^2) - (n+1)E(f) + nE(1) \geq 1.$$

On the other hand,

$$E((f-1)(f-n)) \leq nP(f=0),$$

since all the other terms in the sum have the form $(i-1)(i-n)P(f=i)$ and so are at most 0 (since $(i-1)(i-n) \leq 0$ for $1 \leq i \leq n$). Hence

$$P(f=0) \geq 1/n,$$

as required.

2 The Cycle Index Theorem as stated only allows non-negative integer weights. So let us assume that the weight of a face is a, b or 0 according as its colour is red, blue or white. Then the figure-counting series is $x^a + x^b + 1$, and the function-counting series is

$$Z(G; s_i \leftarrow x^{ai} + x^{bi} + 1).$$

This must hold for all a and b. If, for example, we choose $a > 6b$, then the only way of writing this as a polynomial of degree 6 in $y = x^a$ and $z = x^b$ is as $Z(G; s_i \leftarrow y^i + z^i + 1)$. So this is the required polynomial. In our case, it turns out to be

$$\begin{array}{cccccccccccccc}
 & 1 & + & y & + & 2y^2 & + & 2y^3 & + & 2y^4 & + & y^5 & + & y^6 \\
+ & z & + & 2yz & + & 3y^2z & + & 3y^3z & + & 2y^4z & + & y^5z & & \\
+ & 2z^2 & + & 3yz^2 & + & 6y^2z^2 & + & 3y^3z^2 & + & 2y^4z^2 & & & & \\
+ & 2z^3 & + & 3yz^3 & + & 3y^2z^3 & + & 2y^4z^3 & & & & & & \\
+ & 2z^4 & + & 2yz^4 & + & 2y^2z^4 & & & & & & & & \\
+ & z^5 & + & yz^5 & & & & & & & & & & \\
+ & z^6. & & & & & & & & & & & &
\end{array}$$

3 The answer is just obtained by substituting $s_i \leftarrow r$ in the cycle index of each of the rotation groups in question. For the tetrahedron and the cube these are

$$r^2(r^2+11)/12 \qquad \text{and} \qquad r^2(r+1)(r^3-r^2+4r+8)/24$$

respectively. I leave the others to you!

4 The cycle index of the cyclic group of order 10 is

$$Z(C_{10}) = \frac{1}{10}(s_1^{10}+s_2^5+4s_5^2+4s_{10}),$$

and so substituting $s_i \leftarrow 2$ we find the number to be

$$\frac{1}{10}(2^{10}+2^5+4\cdot 2^2+4\cdot 2) = 108.$$

If we are allowed to turn the necklace over, we obtain five more elements with four 2-cycles and four fixed points (corresponding to turning it about an axis through two opposite beads) and five with five 2-cycles (corresponding to an axis passing midway between opposite pairs of beads). So the cycle index of the *dihedral group* is

$$Z(D_{20}) = \frac{1}{20}(s_1^{10}+6s_2^5+4s_5^2+4s_{10}+5s_1^2s_2^4),$$

and the number of necklaces is

$$\frac{1}{20}(2^{10}+6\cdot 2^5+4\cdot 2^2+4\cdot 2+5\cdot 2^6) = 78.$$

5 (a) This is standard theory of permutation groups. The stabiliser

$$G_x = \{g \in G : x^g = x\}$$

is a subgroup of G, and its right cosets are the sets of the form

$$C(x,y) = \{g \in G : x^g = y\}$$

for each $y \in X$. Now the map $y \mapsto C(x,y)$ is an isomorphism from the given set X to the set of right cosets of G_x in G.

For conjugacy, suppose that the sets of cosets of H and K are isomorphic, and suppose the isomorphism maps the coset H to the coset Ka. Then these two cosets have the same stabiliser: so

$$\{g \in G : Hg = H\} = \{g \in G : Kag = Ka\},$$

so $g \in H$ if and only if $aga^{-1} \in K$, which shows that $a^{-1}Ka = H$, so that H and K are conjugate. Reverse the argument for the converse.

(b) An arbitrary G-space is the disjoint union of transitive G-spaces, and is determined up to isomorphism by the number of transitive G-spaces of each isomorphism type (that is, the number of sstabilisers from each conjugacy class). The counting problem is thus isomorphic to the problem of paying a sum of n units of money using coins whose denominations correspond to the indices of representative subgroups from the conjugacy classes. In other words,

$$\sum_{n\geq 0} d_n(G)t^n = \prod_{i\geq 1}(1-t^i)^{-c_i(G)},$$

from which standard manipulation gives the result:

$$\begin{aligned}
\log\left(\sum_{n\geq 0} d_n(G)t^n\right) &= \sum_{i\geq 1} -c_i(G)\log(1-t^i) \\
&= \sum_{i\geq 1}\sum_{k\geq 1} c_i(G)\frac{t^{ik}}{k} \\
&= \sum_{k\geq 1}\frac{c(t^k)}{k}.
\end{aligned}$$