C50 Enumerative & Asymptotic Combinatorics

Solutions to Exercises 3 Spring 2003

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1 (a) Let G_n be this number. Clearly $G_0 = G_1 = 1$. Also, any expression for n must be either $1+$expression for $n-1$, or $2+$expression for $n-2$; so $G_n = G_{n-1} + G_{n-2}$. By induction, $G_n = F_n$ for all n.

(b) In an expression for n as a sum of ones and twos, suppose there are i twos. There are then $n-2i$ ones, and so $n-i$ terms altogether; the number of such expressions is just the number of choices of i of the $n-i$ positions (to put the twos). Summing over i gives the result.

(c) The generating function for the Fibonacci numbers is

$$\frac{1}{1-x-x^2} = \frac{1+x-x^2}{1-3x^2+x^4}.$$

So

$$\sum_{n\ge1} F_{2n-1}x^{2n-1} = \frac{x}{1-3x^2+x^4}, \qquad \sum_{n\ge0} F_{2n}x^{2n} = \frac{1-x^2}{1-3x^2+x^4},$$

and a change of variable gives

$$\sum_{n\ge1} F_{2n-1}x^{n} = \frac{x}{1-3x+x^2}, \qquad \sum_{n\ge1} F_{2n-2}x^{n} = \frac{x-x^2}{1-3x+x^2}.$$

Now let

$$y = x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2}.$$

The generating function for the numbers calculated in this way is

$$y + y^2 + y^3 + \cdots = \frac{y}{1-y} = \frac{x}{1-3x+x^2},$$

as required.

(d) Let

$$z = x + x^2 + 2x^3 + 4x^4 + \cdots = \frac{x-x^2}{1-2x}.$$

The generating function for the numbers calculated as in this part is

$$z+z^2+z^3+\cdots = \frac{z}{1-z} = \frac{x-x^2}{1-3x+x^2},$$

as required.

(e) Induction on n: for $n=0$ we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

so the result is true. Assuming it for n, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n+3} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{pmatrix} = \begin{pmatrix} F_{n+1} & F_n+F_{n+1} \\ F_n+F_{n+1} & F_{n+1}+F_{n+2} \end{pmatrix},$$

and the result holds using the Fibonacci recurrence.

(f) We need to compute the nth power of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ (this has F_n as the lower right entry). Using the 'square and multiply' method described in the first section of the notes, we see that the required matrix can be found in at most $2\log n$ matrix multiplications, each of which takes six multiplications and three additions (using the fact that we need only calculate three matrix entries).

2 It is clear that the number $f(n)$ satisfies the recurrence

$$f(n) = \sum_{a\in A} f(n-a) \text{ for } n \geq \max(A),$$

and all we have to do is to check the initial conditions. Indeed, we have

$$\left(\sum_{n\geq 0} f(n)x^n\right) \cdot \left(1-\sum_{a\in A} x^a\right) = 1,$$

from which the result follows.

Alternatively,

$$\frac{1}{1-\sum_{a\in A} x^a} = y+y^2+y^3+\cdots,$$

where $y=\sum_{a\in A} x^a$, and coefficient of x^n is clearly $f(n)$.

The smallest root of the polynomial $1-x-x^2-x^5-x^{10}$ is a simple root δ, where δ is approximately 0.584847; so the result holds with $\alpha=\delta^{-1}$, roughly 1.709848.

3 Let $p_a(n)$ be the probability that a doesn't occur in the first n terms, and $q_a(n)$ the probability that it occurs first after n tosses. Then clearly $q_a(n) = p_a(n-1)-p_a(n)$. Hence

$$E_a = \sum nq_a(n) = \sum p_a(n).$$

But $p_a(n) = f_a(n)/2^n$, where $f_a(n)$ is the number of sequences of length n containing no occurrence of a; so this sum of probabilities is equal to $F_a(1/2)$, where $F_a(x) = \sum f_a(n)x^n$. By the theorem of Guibas and Odlyzko (Theorem 1 in the notes),

$$F_a(x) = \frac{C_a(x)}{x^k + (1-2x)C_a(x)}.$$

Putting $x = 1/2$ gives the required probability to be $2^k C_a(1/2)$, as required.

4 (a) In Part 2 of the notes we showed that

$$\sum_{n \geq k} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}.$$

(This is equivalent to the Binomial Theorem for exponent $-(k+1)$.) Replace k by $2k$ and n by $n+k$ to get the result. (Note that we can replace $n \geq 0$ by $n \geq k$ in the question since the terms for $n = 0, \ldots, k-1$ are all zero.)

(b)

$$\begin{aligned}
\sum_{n \geq 0} a_n x^n &= \sum_{k \geq 0} \sum_{n \geq k} \binom{n+k}{2k} 2^{n-k} x^n \\
&= \sum_{k \geq 0} (4x)^{-k} \sum_{n \geq 0} \binom{n+k}{2k} (2x)^{n+k} \\
&= \frac{1}{1-2x} \sum_{k \geq 0} \left(\frac{x}{(1-2x)^2} \right)^k \\
&= \frac{1}{1-2x} \left(1 - \frac{x}{(1-2x)^2} \right)^{-1} \\
&= \frac{1-2x}{1-5x+4x^2},
\end{aligned}$$

as required.

(c)

$$\frac{1-2x}{(1-x)(1-4x)} = \frac{1/3}{1-x} + \frac{2/3}{1-4x},$$

so the coefficient of x^n is $(2^{2n+1}-1)/3$, as required.

Remark It follows that a_n satisfies the recurrence $a_n = 5a_{n-1} - 4a_{n-2}$; but I know of no way to show this directly, or even to suspect from the formula for a_n that it might be true. It's not entirely clear what the 'snake oil method' is; Wilf's explanation of it can be found in *London Math. Soc. Lecture Notes* 141, (1989), 208-217. The term 'snake oil' refers to a worthless quack remedy.

5 There are s_{n-1} such partitions in which $\{n\}$ is a part, and s_{n-2} in which $\{i,n\}$ is a part, for each $i \in \{1,\ldots,n-1\}$. So

$$s_n = s_{n-1} + (n-1)s_{n-2}.$$

The initial values are $s_0 = s_1 = 1$.

Let $S(x) = \sum_{n\geq 0} s_n x^n/n!$. Then

$$\begin{aligned} S'(x) &= \sum_{n\geq 1} \frac{s_n x^{n-1}}{(n-1)!} \\ &= \sum_{n\geq 1} \frac{s_{n-1}x^{n-1}}{(n-1)!} + \sum_{n\geq 2} \frac{(n-1)s_{n-2}x^{n-1}}{(n-1)!} \\ &= (1+x)S(x), \end{aligned}$$

and the solution of this equation with initial condition $S(0) = 1$ is

$$S(x) = \exp\left(x + \frac{x^2}{2}\right).$$

6 For $n = 0$, we have only the empty string. For $n \geq 1$, if a string is non-empty, there are n letters with which it may start, followed by any string made from the remaining $n-1$ letters, so the recurrence holds.

We have

$$\mathrm{e}\,n! = n! \sum_{i=0}^{n} \frac{1}{i!} + n! \sum_{i\geq n+1} \frac{1}{i!};$$

the first term is an integer and the second is less than 1, so

$$\lfloor \mathrm{e}\,n! \rfloor = n! \sum_{i=0}^{n} \frac{1}{i!}.$$

It is easy to check directly that this expression satisfies the recurrence relation and initial condition, and so is equal to a_n. Alternatively, use the method in the notes for recurrence relations of the form $a_n = p_n a_{n-1} + q_n$.

Let

$$\begin{aligned} A(x) &= \sum_{n\geq 0} \frac{a_n x^n}{n!} \\ &= \sum_{n\geq 1} \frac{a_{n-1}x^n}{(n-1)!} + \sum_{n\geq 0} \frac{x^n}{n!} \\ &= xA(x) + \exp(x). \end{aligned}$$

So

$$A(x) = \frac{\exp(x)}{1-x}.$$