C50 Enumerative & Asymptotic Combinatorics

Solutions to Exercises 2 — Spring 2003

1 We construct a bijection between the unordered selections of k objects from a set $\{x_1,\ldots,x_n\}$ with repetitions allowed, and k-element subsets of a set of size $n+k-1$.

Take a line of $n-k+1$ boxes. Suppose first that we are given a k-element subset S of the set of boxes. Place "markers" $a_1,\ldots,a_{n-1}$ in the remaining $n-1$ boxes. Now

- the number of boxes before a_1 is the number of occurrences of x_1 in the selection;
- the number of boxes between a_i and a_{i+1} is the number of occurrences of x_{i+1} in the selection, for $1 \le i \le n-2$;
- the number of boxes after a_{n-1} is the number of occurrences of x_n in the selection.

It is clear that the sum of the numbers of occurrences of the x_i is equal to k.

Conversely, suppose that we have a selection of k of the objects $\{x_1,\ldots,x_n\}$, with repetitions allowed. Let m_i be the number of occurrences of x_i in the selection. Now

- leaving m_1 empty boxes, put marker a_1 in the next box;
- leaving m_i empty boxes after marker a_{i-1}, put marker a_i in the next box, for $2 \le i \le n-1$;
- then m_n empty boxes remain at the end.

The corresponding subset consists of the unmarked boxes.

For example, the subset $\{1,3,4\}$ of $\{1,\ldots,7\}$ corresponds to the selection containing no x_1, one x_2, no x_3, and three x_4.

2 By the Binomial Theorem with $x=y=1$, we have

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Since all terms are positive, each individual term cannot exceed 2^n.

On the other hand,

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k},$$

so we see that

- if $n-k > k+1$ (that is, $k < (n-1)/2$), then $\binom{n}{k} < \binom{n}{k+1}$;
- if $n-k = k+1$ (that is, $k = (n-1)/2$), then $\binom{n}{k} = \binom{n}{k+1}$;
- if $n-k > k+1$ (that is, $k > (n-1)/2$), then $\binom{n}{k} > \binom{n}{k+1}$.

We are given that n is even, so the second alternative doesn't hold; thus the binomial coefficients increase as far as $\binom{n}{n/2}$ and then decrease. So $\binom{n}{n/2}$ is the largest of the $n+1$ terms in the sum, so $\binom{n}{n/2} \geq 2^n/(n+1)$.

Using Stirling's formula we obtain

$$\binom{n}{n/2} = \frac{n!}{((n/2)!)^2} \sim \frac{\sqrt{2\pi n}\,n^n}{\mathrm{e}^n} \cdot \frac{\mathrm{e}^n}{\pi n\,(n/2)^n} = \frac{2^n}{\sqrt{\pi n/2}}.$$

For $n = 10$, we have $\binom{10}{5} = 252$, while $2^n/\sqrt{\pi n/2} = 258.368\ldots$.

3 If we use Stirling's formula in the form $n! \sim Cn^{n+1/2}/\mathrm{e}^n$, then the above estimate would become

$$\binom{n}{n/2} \sim \frac{2}{C}\frac{2^n}{\sqrt{n}},$$

So, in n independent trials with probability $1/2$ of success on each trial, the probability of $n/2$ successes would be

$$\binom{n}{n/2}/2^n \sim \frac{2}{C\sqrt{n}}.$$

Comparison with the Central Limit Theorem shows that $2/C = 1/\sqrt{\pi/2}$, so $C = \sqrt{2\pi}$, as required.

4 For the first equality,

$$\begin{aligned}\sum_m (-1)^{m-k}\binom{n}{m}\binom{m}{k} &= \sum_m (-1)^{m-k}\frac{n!}{(n-m)!\,(m-k)!\,k!} \\ &= \sum_m (-1)^{m-k}\binom{n}{k}\binom{n-k}{m-k}.\end{aligned}$$

In the last expression, m runs from k to n, and so $j = m-k$ runs from 0 to $n-k$, so the expression is

$$\binom{n}{k}\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j} = \binom{n}{k}(1-1)^{n-k} = 0,$$

since $k < n$.

The second equality is proved similarly.

5 The formulation is as follows: If (a_n) and (b_n) are sequences with exponential generating functions $A(x)$ and $B(x)$ respectively, then the following are equivalent:

(a) $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$ for all n;

(b) $B(x) = A(x)\exp(x)$.

Your job to prove it!

6 There is a function from the set of permutations onto the set of partitions, which maps each permutation to its cycle decomposition. Hence $B(n) \leq n!$.

There are many ways to do the first part: here is one requiring not much calculation. Let $s(n)$ be the number of partitions of $\{1,\ldots,n\}$ with all parts of size 2; this is also equal to the number of *involutions*, that is, permutations whose square is the identity. Clearly $s(n) \leq B(n)$. Furthermore, I claim that any permutation is the product of two involutions. It is enough to prove this for cyclic permutations; now generalise the patterns

$$\begin{aligned}(1,2,3,4,5) &= ((2,5)(3,4))\,((1,2)(3,5)),\\ (1,2,3,4,5,6) &= ((2,6)(3,5))\,((1,2)(3,6)(4,5)).\end{aligned}$$

So $n! \leq s(n)^2$. The result follows.

7 This is clear from the proof of Stirling's formula in the notes.

8 For an arbitrary permutation π, let k be the smallest positive integer such that π maps the set $\{1,\ldots,k\}$ to itself. (Thus $k = n$ if and only if π is connected.) Now π is the concatenation of a connected permutation on $\{1,\ldots,k\}$ and an arbitrary permutation on $\{k+1,\ldots,n\}$; and the expression is unique. So

$$n! = \sum_{k=1}^{n} c(k)(n-k)!.$$

This identity shows that in the product

$$\left(1 + \sum_{n\geq 1} n!\,x^n\right) \cdot \left(1 - \sum_{n\geq 1} c(n)x^n\right),$$

the coefficient of x^n is zero for $n > 0$, so the product is 1.

9 In this question, $(-1)^{n-k}$ should read $(-1)^k$ – sorry!

$$(-1)^k \binom{n}{k} = (-1)^k \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{(-n)(-n+1)\cdots(-n+k-1)}{k!} = \binom{-n+k-1}{k}.$$

Proposition 3 asserts (after noting that the terms for $n < k$ are all zero) that

$$\sum_{n\geq k} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}.$$

Dividing by x^k and putting $j = n-k$, we have

$$\sum_{j\geq 0} \binom{k+j}{j} x^j = (1-x)^{-(k+1)}.$$

Now replace x by $-x$. According to the first part of the question, $\binom{k+j}{j}(-1)^j = \binom{-(k+1)}{j}$, and so we have

$$\sum_{j\geq 0} \binom{-(k+1)}{j} x^j = (1+x)^{-(k+1)},$$

as required.

10 Apply Proposition 14 of the notes with $A(x) = x^k/k!$ (so that $a_k = 1$ and $a_i = 0$ for $i \neq k$).

Since $\sum_{k\geq 1} s(n,k) = 0$ for $n \geq 2$, the sum of the left-hand sides is just x; on the right we have

$$\sum_{k\geq 1} \frac{(\log(1+x)^k)}{k!} = \exp(\log(1+x)) - 1 = x.$$

11 We see that

$$n!T(n,k) = \binom{n}{a_1,\ldots,a_k}(a_1-1)!\cdots(1_k-1)!,$$

where the *multinomial coefficient*

$$\binom{n}{a_1,a_2,\ldots,a_k} = \frac{n!}{a_1!\cdots a_k!}$$

is the number of choices of subsets $A_1,\ldots,A_k$ of $\{1,\ldots,n\}$ which partition the set, and $((a_i - 1)!$ is the number of cyclic permutations on a set of size a_i; so the number $n!\,T(n,k)$ is just the number of choices of a permutation of $\{1,\ldots,n\}$ with k cycles together with an ordering of the cycles. That is,

$$n!\,T(n,k) = k!\,|s(n,k)|.$$

12 (a) To show that σ is an equivalence relation:

- $x\;\rho\;x$ and $x\;\rho\;x$, so $x\;\sigma\;x$.
- Symmetry is clear by definition.
- Suppose that $x\;\sigma\;y$ and $y\;\sigma\;z$. Then $x\;\rho\;y$ and $y\;\rho\;z$, so $x\;\rho\;z$; and $z\;\rho\;y$ and $y\;\rho\;x$, so $z\;\rho\;x$. Thus $x\;\sigma\;z$.

Let $[x]$ be the equivalence class of x. Define $[x] \leq [y]$ if $x\;\rho\;y$. This does not depend on the choice of representatives, for if $x' \in [x]$ and $y' \in [y]$, then $x'\;\rho\;x\;\rho\;y\;\rho\;y'$, so $x'\;\rho\;y'$. If $[x] \leq [y]$ and $[y] \leq [x]$, then $x\;\rho\;y$ and $y\;\rho\;x$, so $x\;\sigma\;y$ and $[x] = [y]$; so the relation is antisymmetric. It is clearly transitive and satisfies trichotomy. So it is a total order.

Given an equivalence relation and a total order of its equivalence classes, put $x\,\rho\,y$ if $[x] \leq [y]$; it is easily seen that this is a total preorder. The constructions are mutually inverse.

It follows that a total preorder is uniquely determined by a partition of the point set and a total order of its parts. So the number of total preorders with k equivalence classes is $S(n,k)k!$, and the result follows.

(b) The e.g.f. of the sequence $n!$ is $1/(1-x)$. The result now follows from Proposition 9.

(c) The radius of convergence of the series is $\log 2$; so, if p_n is the number of total preorders on $\{1,\ldots,n\}$, then

$$n!((\log 2)^{-1}-\varepsilon)^n < p_n < n!((\log 2)^{-1}+\varepsilon)^n$$

for all $n \geq n_0(\varepsilon)$.

13 The sum defining the Lah number $L(n,k)$ can be interpreted as the number of choices in the following scheme: choose a permutation of $\{1,\ldots,n\}$ with m cycles, and then a partition of the set of cycles with k parts, and sum over m. This can also be regarded as follows: choose a partition of $\{1,\ldots,n\}$ with k parts, and then choose a permutation of each part. (The total number of cycles is 'summed out'). If the parts are regarded as ordered, the number having parts of size $a_1,\ldots,a_k$ is

$$\frac{n!}{a_1!\cdots a_k!}a_1!\cdots a_k! = n!\,.$$

So the total number is obtained by dividing this by $k!$ (the number of ways of ordering the parts) and multiplying by $\binom{n-1}{k-1}$ (the number of choices of k positive integers with sum n).

To see the last assertion, note that if $a_1+\cdots+a_k = n$, then $b_1+\cdots+b_k = n-k$, where $b_i = a_i - 1$ is non-negative; conversely any k non-negative integers with sum $n-k$ give rise to k positive integers with sum n. Now k non-negative integers with sum $n-k$ correspond to a selection of $n-k$ objects from a set of k, with repetitions allowed and order unimportant (where b_i is the number of times the ith object is chosen); by Question 1 on this sheet, this number is

$$\binom{(n-k)+k-1}{n-k} = \binom{n-1}{k-1}.$$

14 The coefficient of x^n on the left-hand side of

$$\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right)^2 = (1+x)^{2n} = \sum_{k=0}^{2}n\binom{2n}{k}n^k$$

is

$$\sum_{l=0}^{n}\binom{n}{l}\binom{n}{n-l} = \sum_{l=0}^{n}\binom{n}{l}^2,$$

while the coefficient on the right is $\binom{2n}{n}$.

Argue similarly with

$$\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k\right)\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right) = (1-x)^n(1+x)^n = (1-x^2)^n.$$

The coefficient of x^n on the left is

$$\sum_{l=0}^{n}(-1)^l\binom{n}{l}^2,$$

while on the right we have only even powers, so the answer is zero if n is odd, while if n is even it is $(-1)^{n/2}\binom{n}{n/2}$, as claimed.

15 By the Binomial Theorem,

$$(1-4x)^{-1/2}=\sum_{n\geq 0}\binom{-1/2}{n}(-4x)^n.$$

The coefficient of x^n is

$$\frac{(-4)^n(-1)(-3)\cdots(-(2n-1))}{n!\,2\cdots 2}=\frac{(2n)!}{(n!)^2}=\binom{2n}{n}.$$

The coefficient of x^n in the square of this function is

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}.$$

But this is just $(1-4x)^{-1}$, and the coefficient of x^n is 4^n, as required.