C50 Enumerative & Asymptotic Combinatorics

Solutions to Exercises 1 — Spring 2003

---

1 Calculate

$$(1-x)\left(\sum_{n\geq 0} x^n\right).$$

For $n>0$, the coefficient of x^n is $1-1$: we obtain 1 by choosing 1 from the first bracket and x^n from the second, and -1 by choosing $-x$ from the first bracket and x^{n-1} from the second. Clearly the coefficient of x^0 is 1. So the product of the two formal power series is 1.

2 Yes. This is because the operations on formal power series are valid for Taylor series of analytic functions, and two analytic functions are equal if and only if their Taylor series are equal.

3 We have

$$\exp(\log(1+x)) = \sum_{k\geq 0} \frac{(\log(1+x))^k}{k!}.$$

Now the coefficient of x^n in $(\log(1+x))^k$ is obtained by expressing $n = a_1+\cdots+a_k$, where $a_1,\ldots,a_k$ are positive integers, and multiplying the coefficient $(-1)^{a_i-1}/a_i$ of x^{a_i} in $\log(1+x)$ for $i=1,\ldots,k$. The result is $(-1)^{n-k}T(n,k)$. Now multiply this by $1/k!$ and sum over k to get the result.

The analogous statement is: Let $U(n,k)$ be the result of expressing n as the sum of k positive integers in all possible ways, multiplying the reciprocals of the factorials of the integers, and summing. Then

$$\sum_{k=1}^{n} \frac{(-1)^k U(n,k)}{k} = 0$$

for $n>1$.

4 We have

$$\sum_{n\geq 0} \frac{(x+y)^n}{n!} = \sum_{n\geq 0} \frac{x^n}{n!} \cdot \sum_{n\geq 0} \frac{y^n}{n!}.$$

This is an identity between "formal power series in two variables". The term with total degree n on the left is $(x+y)^n/n!$, while on the right it is $\sum_{k=0}^n (x^k/k!)(y^{n-k}/(n-k)!)$. Equating these, and multiplying by $n!$, gives

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

5 Let $c = 1+b$. By the Binomial Theorem, $(1+b)^n$ is the sum of positive terms including the term $\binom{n}{k+1} b^{k+1}$. The ratio of this term to n^k is about $nb^{k+1}/(k+1)!$, which tends to infinity as $n \to \infty$.

Put $m = \log n$. Then $n^\varepsilon = \mathrm{e}^{\varepsilon m}$, and by the preceding part (as $\mathrm{e}^\varepsilon > 1$), the ratio of this to m tends to infinity.

6 (a) Clearly $f(n) = 0$ for odd n. Suppose that $n = 2m$. Take m boxes with room for two numbers in each box. A partition is obtained by putting the numbers $\{1, \ldots, n\}$ into the boxes. There are $n!$ orderings of the numbers. But the same partition is obtained if the order of the numbers in any box is switched, or if the boxes are filled in a different order; so $2^m\, m!$ different orderings correspond to each partition. So the required number is

$$f(2m) = \frac{(2m)!}{2^m\, m!} = 1 \cdot 3 \cdot 5 \cdots (2m-1).$$

(b) The exponential generating function is

$$\begin{aligned} & \sum_{m \geq 0} \frac{1}{(2m)!} \frac{(2m)!}{2^m\, m!} x^{2m} \\ = \quad & \sum_{m \geq 0} \frac{1}{m!} \left(\frac{x^2}{2} \right)^m \\ = \quad & \exp\left(\frac{x^2}{2} \right). \end{aligned}$$

(c) Sorry – there is a mistake in the question: the asymptotic formula should have m in place of n, where $m = n/2$.

By Stirling's formula, if $n = 2m$, then

$$\begin{aligned} f(n) &= \frac{(2m)!}{2^m\, m!} \\ &\sim \sqrt{2\pi 2m} \left(\frac{2m}{\mathrm{e}} \right)^{2m} \Big/ \sqrt{2\pi m} \left(\frac{2m}{\mathrm{e}} \right)^m \\ &= \sqrt{2} \left(\frac{2m}{\mathrm{e}} \right)^m. \end{aligned}$$

7 For $n = 2$, the sequence $00, 01, 11, 10$ is a Gray code. (Here we represent each subset A of $\{1, \ldots, n\}$ by an n-tuple $e_1 \ldots e_n$, where $e_i = 1$ if $i \in A$, $e_i = 0$ otherwise.)

If $s_1, \ldots, s_{2^n}$ is a Gray code for subsets of $\{1, \ldots, n\}$, then the following sequence is clearly a Gray code for subsets of $\{1, \ldots, n+1\}$:

$$s_1 0, \ldots, s_{2^n} 0, s_{2^n} 1, \ldots, s_1 1.$$