C50 Enumerative & Asymptotic Combinatorics

Prize question 3 — Spring 2003

This question is due to Marcio Soares.

1 Prove that

$$\frac{x^{m+1}-1}{x-1} = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k} (-x)^k (1+x)^{m-2k}.$$

Solution More generally,

$$\frac{x^{m+1}-y^{m+1}}{x-y} = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k} (-xy)^k (x+y)^{m-2k}.$$

We have

$$\frac{1}{1-(x+y)t+xyt^2} = \sum_{n=0}^{\infty} ((x+y)t - xyt^2)^n.$$

The term in t^m on the right is obtained from the terms with $n = m-k$, with $0 \le k \le m/2$, by taking $-xyt^2$ from k factors and $(x+y)t$ from the other $m-2k$; so its coefficient is the right-hand side of the given identity.

Now

$$\begin{aligned}
\frac{1}{1-(x+y)t+xyt^2} &= \frac{1}{(1-xt)(1-yt)} \\
&= \frac{x}{x-y} \cdot \frac{1}{1-xt} - \frac{y}{x-y} \cdot \frac{1}{1-yt} \\
&= \frac{x}{x-y} \sum_{m=0}^{\infty} (xt)^m - \frac{y}{x-y} \sum_{m=0}^{\infty} (yt)^m,
\end{aligned}$$

so the coefficient of t^m is

$$\frac{x}{x-y} x^m - \frac{y}{x-y} \cdot y^m = \frac{x^{m+1}-y^{m+1}}{x-y},$$

as required.

Remark Putting x and y equal to the golden ratio and its algebraic conjugate, so that $x+y = -xy = 1$, we deduce the equality of two well-known formulae for the Fibonacci numbers.