Queen Mary
University of London

C50 Enumerative & Asymptotic Combinatorics

Prize question — Spring 2003

The second prize question was the following:

> Let ω be a primitive dth root of unity. Express $\begin{bmatrix} n \\ k \end{bmatrix}_\omega$ in terms of binomial coefficients (whenever you can).

Here is the solution by Pablo Spiga.

Let d be a natural number, and let ω be a primitive dth root of unity in $\mathbb{C}$, i.e. $\omega^d = 1$. Then, if $0 \le a, b \le d-1$, we have

$$\begin{bmatrix} nd+a \\ kd+b \end{bmatrix}_\omega = \binom{n}{k} \begin{bmatrix} a \\ b \end{bmatrix}_\omega.$$

Note that we are assuming that $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]_\omega = 0$ whenever $a < b$.

Solution By induction on a. We have

$$\begin{aligned} 1-\xi^d &= \prod_{i=1}^{d} (\omega^{i-1} - \xi) \\ &= \prod_{i=1}^{d} (\omega^{i-1} \cdot (1 - \omega^{-i+1}\xi)) \\ &= \prod_{i=1}^{d} \omega^{i-1} \cdot \prod_{i=1}^{d} (1 - \omega^{i-1}\xi). \end{aligned}$$

Thus, we get

$$\prod_{i=1}^{nd} (1 + \omega^{i-1}(-\xi)) = \sum_{j=0}^{nd} \omega^{j(j-1)/2} (-1)^j \begin{bmatrix} nd \\ j \end{bmatrix}_\omega \xi^j, \qquad (1)$$

but

$$\prod_{i=1}^{nd}(1+\omega^{i-1}(-\xi)) = (1-\xi^d)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\xi^{kd}. \tag{2}$$

We have proved that $\left[{nd \atop j}\right]_\omega = 0$ if d does not divide j. Assume $j = dk$. By (1) and (2), as

$$\omega^{dk(dk-1)/2}(-1)^{k(d+1)} = 1, \tag{3}$$

we get

$$\begin{bmatrix} nd \\ kd \end{bmatrix}_\omega = \binom{n}{k}.$$

(For (3), note that if d is odd then $-1^{d+1} = 1$, while if d is even than we can write -1 as $\omega^{d/2}$, and we find $\omega^{dk(dk+d)/2} = \omega^{d^2k(k+1)/2}$.) This proves the result for $a = 0$.

Assume $a \geq 1$. If $b \neq 0$ then, by induction hypothesis and by the usual recurrence relation, we get

$$\begin{aligned} \begin{bmatrix} nd+a \\ kd+b \end{bmatrix}_\omega &= \begin{bmatrix} nd+a-1 \\ kd+b-1 \end{bmatrix}_\omega + \omega^{kd+b}\begin{bmatrix} nd+a-1 \\ kd+b \end{bmatrix}_\omega \\ &= \binom{n}{k}\begin{bmatrix} a-1 \\ b-1 \end{bmatrix}_\omega + \omega^b\binom{n}{k}\begin{bmatrix} a-1 \\ b \end{bmatrix}_\omega \\ &= \binom{n}{k}\begin{bmatrix} a \\ b \end{bmatrix}_\omega. \end{aligned}$$

Finally, if $b = 0$, then, as $a-1 < d-1$,

$$\begin{aligned} \begin{bmatrix} nd+a \\ kd \end{bmatrix}_\omega &= \begin{bmatrix} nd+a-1 \\ (k-1)d+d-1 \end{bmatrix}_\omega + \omega^{kd}\begin{bmatrix} nd+a-1 \\ kd \end{bmatrix}_\omega \\ &= \binom{n}{k}\omega^0\begin{bmatrix} a-1 \\ 0 \end{bmatrix}_\omega \\ &= \binom{n}{k}\begin{bmatrix} a \\ b \end{bmatrix}_\omega. \end{aligned}$$

Remark Compare Lucas' formula

$$\binom{np+a}{kp+b} \equiv \binom{n}{k}\binom{a}{b} \pmod{p}$$

if p is prime and $0 \leq a, b < p$.