Queen Mary
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C50 Enumerative & Asymptotic Combinatorics

Stirling and Lagrange **Spring 2003**

This section of the notes contains proofs of Stirling's formula and the Lagrange Inversion Formula.

Stirling's formula

Theorem 1 (Stirling's Formula)

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Proof Consider the graph of the function $y = \log x$ between $x = 1$ and $x = n$, together with the piecewise linear functions shown in Figure 1.

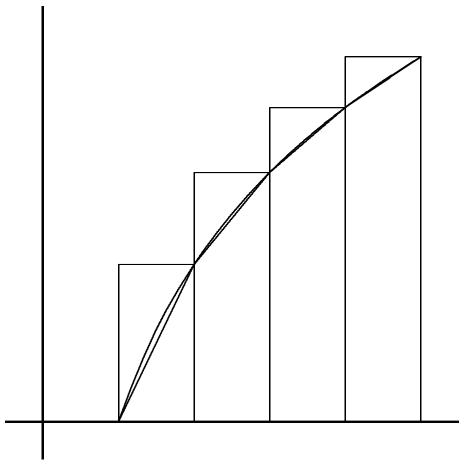

Figure 1: Stirling's formula

Let $f(x) = \log x$, let $g(x)$ be the function whose value is $\log m$ for $m \le x < m+1$, and let $h(x)$ be the function defined by the polygon with vertices $(m, \log m)$, for $1 \le m \le n$. Clearly

$$\int_1^n g(x)\,\mathrm{d}x = \log 2 + \cdots + \log n = \log n!\,.$$

The difference between the integrals of g and h is the sum of the areas of triangles with base 1 and total height $\log n$; that is, $\frac{1}{2}\log n$.

Some calculus[1] shows that the difference between the integrals of f and g tends to a finite limit c as $n \to \infty$.

Finally, a simple integration shows that

$$\int_1^n f(x)\,\mathrm{d}x = n\log n - n + 1.$$

We conclude that

$$\log n! = n\log n - n + \tfrac{1}{2}\log n + (1-c) + o(1),$$

so that

$$n! \sim \frac{Cn^{n+1/2}}{\mathrm{e}^n}.$$

To identify the constant C, we can proceed as follows. Consider the integral

$$I_n = \int_0^{\pi/2} \sin^n x\,\mathrm{d}x.$$

Integration by parts shows that

$$I_n = \frac{n-1}{n} I_{n-2},$$

[1]Let $F(x) = f(x) - g(x)$. The convexity of $\log x$ shows that $F(x) \ge 0$ for all $x \in [m, m+1]$. For an upper bound we use the fact, a consequence of Taylor's Theorem, that

$$\log x \le \log m + \frac{x-m}{m} \le \log m + \frac{1}{m}$$

for $x \in [m, m+1]$. Then

$$F(x) = \log x - \log m - \log\left(1 + \frac{1}{m}\right)(x-m) \le \frac{1}{m} - \log\left(1 + \frac{1}{m}\right) \le \frac{1}{2m^2},$$

where the last inequality comes from another application of Taylor's Theorem which yields $\log(1+x) \ge x - x^2/2$ for $x \in [0,1]$. Now $\sum(1/m^2)$ converges, so the integral is bounded.

and hence

$$\begin{aligned} I_{2n} &= \frac{(2n)!\,\pi}{2^{2n+1}(n!)^2}, \\ I_{2n+1} &= \frac{2^{2n}(n!)^2}{(2n+1)!}. \end{aligned}$$

On the other hand,

$$I_{2n+2} \le I_{2n+1} \le I_{2n},$$

from which we get

$$\frac{(2n+1)\pi}{4(n+1)} \le \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!} \le \frac{\pi}{2},$$

and so

$$\lim_{n\to\infty} \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!} = \frac{\pi}{2}.$$

Putting $n! \sim Cn^{n+1/2}/\mathrm{e}^n$ in this result, we find that

$$\frac{C^2\mathrm{e}}{4} \lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^{-(2n+3/2)} = \frac{\pi}{2},$$

so that $C = \sqrt{2\pi}$.

Lagrange Inversion

A formal power series over a field, with zero constant term and non-zero term in x, has an inverse with respect to composition. The associative, closure, and identity laws are obvious, and the rule for finding the inverse in characteristic zero is known as *Lagrange inversion*. We work over $\mathbb{R}$ for convenience.

The theorem

The basic fact can be stated as follows.

Proposition 2 *Let f be a formal power series over $\mathbb{R}$, with $f(0) = 0$ and $f'(0) \neq 0$. Then there is a unique formal power series g such that $g(f(x)) = x$; the coefficient of y^n in $g(y)$ is*

$$\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left(\frac{x}{f(x)}\right)^n\right]_{x=0} \Big/ n!.$$

This can be expressed in a more convenient way for our purpose. Let

$$\phi(x) = \frac{x}{f(x)}.$$

Then the inverse function g is given by the functional equation

$$g(y) = y\phi(g(y)).$$

Then Lagrange inversion has the form

$$g(y) = \sum_{n\geq 1} \frac{b_n y^n}{n!},$$

where

$$b_n = \left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\phi(x)^n\right]_{x=0}.$$

Example: Cayley's Theorem The exponential generating function for rooted trees satisfies the equation

$$T^*(x) = x\exp(T^*(x)).$$

With $\phi(x) = \exp(x)$, we find that the coefficient of $y^n/n!$ in $T^*(y)$ is

$$\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\exp(nx)\right]_{x=0} = n^{n-1}.$$

Now there are n ways to root a given tree; so the number of trees is n^{n-2}, proving Cayley's Theorem.

Proof of the theorem

The proof of Lagrange's inversion formula involves a considerable detour. The treatment here follows the book by Goulden and Jackson. Throughout this section, we assume that the coefficients form a field of characteristic zero; for convenience, we assume that the coefficient ring is $\mathbb{R}$.

First, we extend the notion of formal power series to *formal Laurent series*, defined to be a series of the form

$$f(x) = \sum_{n\geq m} a_n x^n,$$

where m may be positive or negative. if the series is not identically zero, we may assume without loss of generality that $a_m \neq 0$, in which case m is the *valuation* of f, written

$$m = \mathrm{val}(f).$$

We define addition, multiplication, composition, differentiation, etc., for formal Laurent series as for formal power series. In particular, $f(g(x))$ is defined for any formal Laurent series f, g with $\mathrm{val}(g) > 0$. (This is less trivial than the analogous result for formal power series. In particular, we need to know that $g(x)^{-m}$ exists as a formal Laurent series for $m > 0$. It is enough to deal with the case $m = 1$, since certainly $g(x)^m$ exists. If $\mathrm{val}(g) = r$, then $g(x) = x^r g_1(x)$, and so $g(x)^{-1} = x^{-r} g_1(x)^{-1}$, and we have seen that $g_1(x)^{-1}$ exists as a formal power series, since $g_1(0)$ is invertible.

We denote the derivative of the formal Laurent series $f(x)$ by $f'(x)$.

We also introduce the following notation: $[x^n]f(x)$ denotes the coefficient of x^n in the formal power series (or formal Laurent series) $f(x)$. The case $n = -1$ is especially important, as we learn from complex analysis. The value of $[x^{-1}]f(x)$ is called the *residue* of $f(x)$, and is also written as $\mathrm{Res}\, f(x)$.

Everything below hinges on the following simple observation, which is too trivial to need a proof.

Proposition 3 *For any formal Laurent series $f(x)$, we have* $\mathrm{Res}\, f'(x) = 0$.

Now the following result describes the residue of the composition of two formal Laurent series.

Theorem 4 (Residue Composition Theorem) *Let $f(x)$, $g(x)$ be formal Laurent series with* $\mathrm{val}(g) = r > 0$. *Then*

$$\mathrm{Res}(f(g(x))g'(x)) = r\,\mathrm{Res}(f(x)).$$

Proof It is enough to consider the case where $f(x) = x^n$, since Res is a linear function.

Suppose that $n \neq -1$, so that the right-hand side is zero. Then

$$\mathrm{Res}(g^n(x)g'(x)) = \frac{1}{n+1}\mathrm{Res}\left(\frac{\mathrm{d}}{\mathrm{d}x} g^{n+1}(x)\right) = 0.$$

So consider the case where $n = -1$. Let $g(x) = ax^r h(x)$, where $a \neq 0$ and $h(0) = 1$. Then

$$\begin{aligned} g'(x)g(x)^{-1} &= \frac{\mathrm{d}}{\mathrm{d}x}\log g(x) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(\log a + r\log x + \log h(x)) \\ &= \frac{r}{x} + \frac{\mathrm{d}}{\mathrm{d}x}\log h(x), \end{aligned}$$

so

$$\mathrm{Res}\, g'(x)g(x)^{-1} = r = r\,\mathrm{Res}\, x^{-1}.$$

Note that we have cheated slightly in the first line of this argument: $\log g(x)$ may not exist as a formal Laurent series; but it is the case that for if the equation $f(x) = g(x)h(x)$ holds for formal Laurent series, then

$$f'(x)/f(x) = g'(x)/g(x) + h'(x)/h(x),$$

and in the preceding argument it is the case that $\log h(x)$ exists (this is obtained by substituting $y = h(x) - 1$ in $\log(1+y)$) and its derivative is $h'(x)/h(x)$. Consider this point carefully; an error here would lead to the incorrect conclusion that $\mathrm{Res}(g'(x)/g(x)) = 0$.

From the Residue Composition Theorem, we can prove a more general version of Lagrange Inversion.

Theorem 5 (Lagrange Inversion) *Let* ϕ *be a formal power series with* $\mathrm{val}(\phi) = 1$, *Then the equation*

$$g(x) = x\phi(g(x)$$

has a unique formal power solution $g(x)$. *Moreover, for any Laurent series* f, *we have*

$$[x^n]\, f(g(x)) = \begin{cases} \frac{1}{n}[x^{n-1}](f'(x)\phi(x)^n) & \text{if } n \geq \mathrm{val}(f) \text{ and } n \neq 0, \\ f(0) + \mathrm{Res}(f'(x)\log(\phi(0)^{-1}\phi(x)) & \text{if } n = 0. \end{cases}$$

Proof Let $\Phi(x) = x/\phi(x)$, so that $\Phi(g(x)) = x$ and $\mathrm{val}(\Phi(x)) = 1$. Then g is the inverse function of Φ.

We have

$$\begin{aligned} [x^n]\, f(g(x)) &= \mathrm{Res}\, x^{-n-1} f(g(x)) \\ &= \mathrm{Res}\, \Phi(y)^{-n-1}\Phi'(y)\, f(y), \end{aligned}$$

where we have made the swubstitution $x = \Phi(y)$ (so that $y = g(x)$) and used the Residue Composition Theorem.

For $n \neq 0$, we have

$$\begin{aligned} [x^n]\, f(g(x)) &= -\frac{1}{n}[y^{-1}]\, f(y) \left(\Phi(y)^{-n}\right)' \\ &= \frac{1}{n}[y^{-1}]\, f'(y)\Phi(y)^{-n} \\ &= \frac{1}{n}[y^{n-1}]\, f'(y)\phi(y)^n. \end{aligned}$$

Here, in the second line, we have used the fact that

$$\mathrm{Res}(f'(x)g(x)) = -\mathrm{Res}(f(x)g'(x)),$$

a consequence of the fact that $\mathrm{Res}(f(x)g(x))' = 0$; in the third line we use the fact that $\Phi(x) = x/\phi(x)$.

For $n = 0$, we have

$$\begin{aligned}[x^0]f(g(x)) &= [y^0]f(y) - [y^{-1}]f(y)\phi'(y)\phi(y)^{-1} \\ &= f(0) + \mathrm{Res}(f'(y)\log(\phi(y)\phi^{-1}(0)),\end{aligned}$$

using the same principle as before and the fact that

$$(\log(\phi(y)\phi^{-1}(0)))' = \phi'(y)\phi(y)^{-1}.$$

Taking $f(x) = x$ in this result gives the form of Lagrange Inversion quoted earlier.

We proceed to an application, also taken from Goulden and Jackson, of the Residue Composition Theorem.

Example: a binomial identity We use the Residue Composition Theorem to prove that

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1}\binom{j+k}{2n} = \binom{2j}{2n}.$$

We begin with the sum of the odd terms in $(1+x)^{2n+1}$:

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} x^{2k} = \frac{1}{2x}\left((1+x)^{2n+1} - (1-x)^{2n+1}\right).$$

Call the right-hand side of this equation $f(x)$. Now, if S is the sum that we want to evaluate, then

$$\begin{aligned} S &= [y^{2n}](1+y)^j \sum_{k=0}^{n} \binom{2n+1}{2k+1}(1+y)^k \\ &= \mathrm{Res}\, y^{-(2n+1)}(1+y)^j f((1+y)^{1/2}). \end{aligned}$$

Now we do the following rather strange thing: make the substitution $y = z^2(z^2-2)$. Then $\mathrm{val}(y(z)) = 2$, and $(1+y)^{1/2} = 1 - z^2$. So the Residue Composition Theorem gives

$$\begin{aligned} S &= \mathrm{Res}(z^2-1)^{2j}\left(\frac{1}{(z^2-2)^{2n+1}} - \frac{1}{z^{4n+2}}\right) z \\ &= \mathrm{Res}(z^2-1)^{2j} z^{-(4n+1)} \\ &= [z^{4n}](z^2-1)^{2j} \\ &= \binom{2j}{2n}, \end{aligned}$$

as required. (In the second line we have used the fact that $(z^2-2)^{-(2n+1)}$ is a formal power series and so its residue is zero.)