Queen Mary
University of London

C50 Enumerative & Asymptotic Combinatorics

Notes 5 **Spring 2003**

A cube has six faces, so if we paint each face red, white or blue, the total numbers of ways that we can apply the colours is $3^6 = 729$. However, if we can pick up the cube and move it around, it is natural to count in a different way, where two coloured cubes differing only by a rotation are counted as "the same". There are 24 rotations of the cube into itself, but the answer to our question is not obtained just by dividing 729 by 24. The purpose of this section is to develop tools for answering such questions.

Group actions

Let X be a set, and G a set of permutations of X. We write the image of $x \in X$ under the permutation g as x^g. We denote the identity permutation (leaving every element of X where it is) by 1, and the inverse of a permutation g (the permutation h with $x^g = y \Leftrightarrow x^h = x$) by g^{-1}. The composition of two permutations g and h, denoted by gh, is defined by the rule that

$$x^{gh} = (x^g)^h$$

(in other words, apply first g, then h).

We say that G is a *permutation group* if the following conditions hold:

- G contains the identity permutation;
- G contains the inverse of each of its elements;
- G contains the composition of any two of its elements.

For example, the 24 rotational symmetries of a cube form a permutation group on the set of points of the cube.

Until the middle of the nineteenth century, what we have just defined would have simply been called a *group*. Now the definition of a group is more abstract. We don't go into abstract group theory here, but note some terminology arising from this. If G is an abstract group in the modern sense, an *action* of G on the set X is a function associating a permutation of X with each group element, in such a way that the identity, inverse, and composition of permutations correspond to the same concepts in the abstract group.

In particular, if G is a permutation group on a set X, then we can construct actions of G on various auxiliary sets built from X: for example, the set of ordered pairs of elements of X, the set of subsets of X, the set of functions from X to another set (or from another set to X).

For example, G acts on the set $X \times X$ of ordered pairs of elements of X by the rule

$$(x,y)^g = (x^g, y^g)$$

for $x,y \in X$, $g \in G$; that is, the permutation g acts coordinate-wise on ordered pairs, mapping (x,y) to (x^g, y^g).

Thus, the phrases "G is a permutation group on X" and "G acts on X" are almost synonymous; the difference is of less interest to a combinatorialist than to an algebraist.

Suppose that G acts on X. We define a relation $\sim$ on X by the rule that $x \sim y$ if $y = x^g$ for some $g \in G$.

Proposition 1 *$\sim$ is an equivalence relation.*

Proof We check the three conditions.

- $x = x^1$, so $x \sim x$: $\sim$ is reflexive.
- Let $x \sim y$. Then $y = x^g$, so $x = y^{g^{-1}}$, so $y \sim x$: $\sim$ is symmetric.
- Let $x \sim y$ and $y \sim z$. Then $x = x^g$ and $z = y^h$, for some $g,h \in G$. Thus, $z = (x^g)^h = x^{gh}$, so $x \sim z$: $\sim$ is transitive.

Note that the three conditions in the definition of a permutation group translate precisely into the three conditions of an equivalence relation.

The equivalence classes of this relation are the *orbits* of G on X.

In our coloured cube example, the group of 24 rotations of the cube acts on the set of 729 colourings of the faces of the cube. Two colourings count "the same" if and only if they are in the same orbit. So our task is to count orbits.

The Orbit-Counting Lemma

For any permutation g of X, we let $\mathrm{fix}(g)$ denote the number of *fixed points* of g (elements $x \in X$ such that $x^g = x$).

Theorem 2 ***(Orbit-Counting Lemma)*** *Let G be a permutation group on the finite set X. Then the number of orbits of G on X is given by the formula*

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{fix}(g).$$

Proof We count in two different ways the number N of pairs (x,g), with $x \in X$, $g \in G$, and $x^g = x$.

On the one hand, clearly

$$N = \sum_{g\in G}\mathrm{fix}(g).$$

On the other hand, we claim that if the point x lies in an orbit $\{x = x_1, \ldots, x_n\}$, then the number of permutations $g \in G$ with $x^g = x$ is $|G|/n$. More generally, for any i with $1 \le i \le n$, the number of permutations $g \in G$ with $x^g = x_i$ is independent of i (the proof is an exercise), and so is $|G|/n$.

Hence the number of pairs (y,g) with $y^g = y$ for which y lies in a fixed orbit of size n is $n \cdot |G|/n = |G|$. So each orbit contributes $|G|$ to the sum, and so $N = |G|k$, where k is the number of orbits.

Equating the two values gives the result.

Using this, we can count our coloured cubes. WE have to examine the 24 rotations and find the number of colourings fixed by each.

- The identity fixes all $3^6 = 729$ colourings.
- There are three axes of rotation through the mid-points of opposite faces. A rotation through a half-turn about such an axis fixes $3^4 = 81$ colourings: we can choose arbitrarily the colour for the top face, the bottom face, the east and west faces, and the north and south faces (assuming that the axis is vertical). A rotation about a quarter turn fixes $3^3 = 27$ colourings, since all four faces except top and bottom must have the same colour. There are three half-turns and six quarter-turns.
- A half-turn about the axis joining the midpoints of opposite edges fixes $3^3 = 27$ colourings. There are six such rotations.

- A third-turn about the axis joining opposite vertices fixes $3^2 = 9$ colourings. There are eight such rotations.

By Theorem 2, the number of orbits is

$$\frac{1}{24}(1 \cdot 729 + 3 \cdot 81 + 6 \cdot 27 + 6 \cdot 27 + 8 \cdot 9) = 57,$$

so there are 57 different colourings up to rotation.

At this point, we can give a more combinatorial proof of the formula

$$x(x-1)\cdots(x-n+1) = \sum_{k=1}^{n} s(n,k)x^k$$

from chapter 2. We prove the equivalent form

$$x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n} |s(n,k)|x^k$$

from which the required equation is obtained by substituting $-x$ for x and multiplying by $(-1)^n$. Suppose first that x is a positive integer. Consider the set of functions from $\{1,\ldots,n\}$ to a set X of cardinality x. There are x^n such functions. Now the symmetric group S_n acts on these functions: the permutation g maps the function f to f^g, where

$$f^g(i) = f(ig^{-1}).$$

The orbits are simply the selections of n things from X, where repetitions are allowed and order is not important. So the number of orbits is

$$\binom{x+n-1}{n} = x(x+1)\cdots(x+n-1)/n!$$

(see Exercise 2.1).

We can also count the orbits using the Orbit-Counting Lemma. Let g be a permutation in S_n having k cycles. How many functions are fixed by g? Clearly a function f is fixed if and only if it is constant on each cycle of g; its values on the cycles can be chosen arbitrarily. So there are x^k fixed functions. Since the number of permutations with k cycles is $|s(n,k)|$, the Orbit-Counting Lemma shows that the number of orbits is

$$\frac{1}{n!}\sum_{k=1}^{n} |s(n,k)|x^k.$$

Equating the two expressions and multiplying by $n!$ gives the result.

Now the required equation holds for all positive integer values of x, and so it is a polynomial identity.

Cycle index

It is possible to develop a method for solving the coloured cubes problem which doesn't require extensive recalculation when small changes are made (such as changing the number of colours).

Suppose that we have a set F of objects called "figures", each of which (say f) has a non-negative integer "weight" $w(f)$ associated with it. The number of figures may be infinite, but we assume that there are only a finite number of any given weight: say a_n figures of weight n. The *figure-counting series* is the (ordinary) generating function for these numbers:

$$A(x) = \sum_{n\geq 0} a_n x^n.$$

We attach a figure to each point of a finite set X. (Equivalently, we take a function ϕ from X to the set F of figures.) The *weight* of the function ϕ is just

$$w(\phi) = \sum_{x\in X} w(\phi(x)).$$

Finally, we have a group G of permutations of X. Then G acts on the set of functions by the rule that

$$\phi^g(x) = \phi(xg^{-1}).$$

Clearly $w(\phi^g) = w(\phi)$ for any function ϕ.

We want to find the generating function for the number of functions of each possible weight, but counting two functions as "the same" if they lie in the same orbit of G with the above action. In other words, we want to calculate the *function-counting series*

$$B(x) = \sum_{n\geq 0} b_n x^n,$$

where b_n is the number of orbits consisting of functions of weight n.

In the coloured cubes example, if we take three figures Red, White and Blue, each of weight 0, the figure-counting series is simply 3, and the function-counting series is 57. We could, say, change the weight of Red to 1, so that the figure-counting series is $2+x$; then the function-counting series is the generating function for the numbers of colourings with $0,1,2,\ldots,6$ red faces (up to rotations).

The gadget that does this job is the *cycle index* of G. Each element $g \in G$ can be decomposed into disjoint cycles; let $c_i(g)$ be the number of cycles of length i, for $i = 1,\ldots,n = |X|$. Now put

$$z(g) = s_1^{c_1(g)} s_2^{c_2(g)} \cdots s_n^{c_n(g)},$$

where $s_1, \dots, s_n$ are indeterminates. Then the *cycle index* of G is defined to be

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} z(g).$$

For example, our analysis of the rotations of the cube shows that the cycle index of this group (acting on faces) is

$$\frac{1}{24}(s_1^6 + 3s_1^2 s_2^2 + 6s_1^2 s_4 + 6s_2^3 + 8s_3^2).$$

We use the notation

$$Z(G; s_i \leftarrow f_i \text{ for } i = 1, \dots, n)$$

for the result of substituting the expression f_i for the indeterminate s_i for $i = 1, \dots, n$.

Theorem 3 *If G acts on X, and we attach figures to the points of X with figure-counting series $A(x)$, then the function-counting series is given by*

$$B(x) = Z(G; s_i \leftarrow A(x^i) \textit{ for } i = 1, \dots, n).$$

For example, in the coloured cubes, let Red have weight 1 and the other colours weight 0. Then $A(x) = 2 + x$, and the function-counting series is

$$\begin{aligned} B(x) &= \frac{1}{24}((2+x)^6 + 3(2+x)^2(2+x^2)^2 + 6(2+x)^2(2+x^4) \\ &\qquad + 6(2+x^2)^3 + 8(2+x^3)^2) \\ &= 10 + 12x + 16x^2 + 10x^3 + 6x^4 + 2x^5 + x^6. \end{aligned}$$

Note that putting $x = 1$ recovers the value 57.

Proof The first step is to note that, if we ignore the group action and simply count all the functions, the function-counting series is $B(x) = A(x)^n$, where $n = |X|$. For the term in x^m in $A(x)^n$ is obtained by taking all expressions $m = m_1 + \cdots + m_n$ for m as a sum of n non-negative integers, multiplying the corresponding terms $a_{m_i}^{m_i}$ in $A(x)$, and summing the result. The indicated product counts the number of choices of functions of weights $m_1, \dots, m_n$ to attach at the points $1, \dots, n$ of X, so the result is indeed the function-counting series.

Note that this proves the theorem in the case where G is the trivial group.

Next, we have to count the functions of given weight fixed by a permutation $g \in G$. As we have seen, a function is fixed by g if and only if it is constant on the cycles of g.

Now if we choose a function of weight r to attach to the points of a particular i-cycle of g, the number of choices is a_r but the contribution to the weight is ir. Arguing as above, the generating function for the number of fixed functions is

$$A(x)^{c_1(g)}A(x^2)^{c_2(g)}\cdots A(x^n)^{c_n(g)} = z(g; s_i \leftarrow A(x^i) \text{ for } i = 1, \ldots, n).$$

Finally, by the Orbit-Counting Lemma, if we sum over $g \in G$ and divide by $|G|$, we find that the function-counting series is

$$B(x) = Z(G; s_i \leftarrow A(x^i) \text{ for } i = 1, \ldots, n).$$