C50 Enumerative & Asymptotic Combinatorics

Notes 3 Spring 2003

A recurrence relation expresses the nth term of a sequence as a function of the preceding terms. The most general form of a recurrence relation takes the form

$$x_n = F_n(x_0, \ldots, x_{n-1}) \text{ for } n \geq 0.$$

Clearly such a recurrence has a unique solution. (Note that this allows the possibility of prescribing some initial values, by choosing the first few functions to be constant.)

In general there is no hope of "solving" such a relation. There is a small class of relations whith can be solved systematically, and a larger class which can be solved by trickery.

Example: Ordered number partitions In how manny ways is it possible to write the positive integer n as a sum of positive integers, where the order of the summands is significant?

Let x_n be this number. One possible expression has a single summand n. In any other expression, if $n-i$ is the first summand, then it is followed by an expression for i as an ordered sum, of which there are x_i possibilities. Thus

$$x_n = 1 + x_1 + x_2 + \cdots + x_{n-1},$$

for $n \geq 1$. (When $n = 1$, this reduces to $x_1 = 1$.)

Since

$$x_{n-1} = 1 + x_1 + x_2 + \cdots + x_{n-2},$$

the recurrence reduces to the much simpler form

$$x_n = 2x_{n-1} \text{ for } n > 1,$$

with initial condition $x_1 = 1$. This obviously has the solution $x_n = 2^{n-1}$ for $n \geq 1$.

Linear recurrences with constant coefficients

Bounded recurrences

One type of linear recurrence which can be solved completely is of the form

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k} \tag{1}$$

for $n \geq k$, where the k values $x_0, x_1, \ldots, x_{k-1}$ are prescribed.

If we consider the recurrence (1) without the initial values, we see that sums and scalar multiples of solutions are solutions. So, taking sequences over a field such as the rational numbers, we see that the set of solutions is a vector space over the field. Its dimension is k, since the k initial values can be prescribed abitrarily.

Thus, if we can write down k linearly independent solutions, the general solution is a linear combination of them.

The *characteristic equation* of the recurrence (1) is the equation

$$x^k - a_1 x^{k-1} - \cdots - a_k = 0.$$

This polynomial has k roots, some of which may be repeated. Suppose that its distinct roots are $\alpha_1, \ldots, \alpha^r$ with multiplicities $m_1, \ldots, m_r$, where $m_1 + \cdots + m_r = k$. Then a short calculation shows that the k functions

$$x_n = \alpha_1^n, \ldots, n^{m_1-1}\alpha_1^n, \ldots, \alpha_r^n, \ldots, n^{m_r-1}\alpha_r^n$$

are solutions of (1); they are clearly linearly independent. So the general solution is a linear combination of them.

Example: Fibonacci numbers Consider the Fibonacci recurrence

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

The characteristic equation is

$$x^2 - x - 1 = 0$$

with roots $\alpha, \beta = (1 \pm \sqrt{5})/2$. So the general solution is

$$F_n = A\alpha^n + B\beta^n,$$

and A and B can be determined from the initial conditions.

For the usual Fibonacci numbers, we have $F_0 = F_1 = 1$, giving the two equations

$$\begin{aligned} A + B &= 1, \\ A\alpha + B\beta &= 1. \end{aligned}$$

Solving these equations gives the solution we found earlier.

Example: Sequences with forbidden subwords Let a be a binary sequence of length k. How many binary sequences of length n do not contain a as a consecutive subword?

Suppose, for example, that $a = 11$, so that we are counting sequences with no two consecutive ones. Let $f(n)$ denote the number of such sequences of length n, and $g(n)$ the number of these commencing with 1. Then

$$\begin{aligned} f(n) &= 2f(n-1) - g(n-1), \\ g(n) &= f(n-1) - g(n-1), \end{aligned}$$

since a sequence commencing with 0 can be preceded with either 0 or 1, while a sequence commencing with 1 can only be preceded with 0. A little manipulation gives

$$f(n) = f(n-1) + f(n-2),$$

the Fibonacci recurrence relation. Since $f(1) = 2 = F_2$ and $f(2) = 3 = F_3$, we conclude that $f(n) = F_{n+1}$, the $(n+1)$st Fibonacci number.

Guibas and Odlyzko extended this approach to arbitrary forbidden substrings. They defined the *correlation polynomial* of a binary string a of length k to be

$$C_a(x) = \sum_{j=0}^{k-1} c_a(j)x^j,$$

where $c_a(0) = 1$ and, for $1 \leq j \leq k-1$,

$$c_a(j) = \begin{cases} 1 & \text{if } a_1a_2\cdots a_{k-j} = a_{j+1}a_{j+2}\cdots a_k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $a = 11$, we have $C_a(x) = 1 + x$.

Theorem 1 *Let $f_a(n)$ be the number of binary strings of length n excluding the substring a of length k. Then the generating function $F_a(x) = \sum_{n\geq 0} f_a(n)x^n$ is given by*

$$F_a(x) = \frac{C_a(x)}{x^k + (1-2x)C_a(x)},$$

where $C_a(x)$ is the correlation polynomial of a.

Proof We define $g_a(n)$ to be the number of binary sequences of length n which commence with a but have no other occurrence of a as a consecutive subsequence, and $G_a(x) = \sum_{n\geq 0} g_a(n)x^n$ the generating function of this sequence of numbers.

Let b be a sequence counted by $f_a(n)$. Then for $x \in \{0,1\}$, the sequence xb contains a at most once at the beginning. So

$$2f_a(n) = f_a(n+1) + g_a(n+1).$$

Multiplying by x^n and summing over $n \geq 0$ gives

$$2F_a(x) = x^{-1}(F_a(x) - 1 + G_a(x)). \tag{2}$$

Now let c be the concatenation ab. Then c starts with a, and may contain other occurrences of a, but only at positions overlapping the initial a, that is, where $a_{k-j+1}\cdots a_k b_1 \cdots b_j = a_1 \cdots a_k$. This can only occur when $c_a(k-j) = 1$, and the sequence $a_{k-j+1}\cdots a_k b$ then has length $n+j$ and has a unique occurrence of a at the beginning. So

$$f_a(n) = \sum g_a(n+j),$$

where the sum is over all j with $1 \leq j \leq k$ for which $c_a(k-j) = 1$. This can be rewritten

$$f_a(n) = \sum_{j=1}^{k} c_a(k-j) g_a(n+j),$$

or in terms of generating functions,

$$F_a(x) = x^{-k} C_a(x) G_a(x). \tag{3}$$

Combining equations (2) and (3) gives the result.

In the case where $a = 11$, we obtain

$$F_{11}(x) = \frac{1+x}{x^2 + (1-2x)(1+x)} = \frac{1+x}{1-x-x^2},$$

so that $f_{11}(n) = F_n + F_{n-1} = F_{n+1}$, as previously noted.

Unbounded recurrences

We will give here just one example. Recall from the last chapter that the generating function for the number $p(n)$ of partitions of the integer n is given by

$$\sum_{n\geq 0} p(n)x^n = \left(\prod_{k\geq 1}(1-x^k)\right)^{-1}.$$

Thus, to get a recurrence relation for $p(n)$, we have to understand the coefficients of its inverse:

$$\sum_{n\geq 0} a(n)x^n = \prod_{k\geq 1}(1-x^k).$$

Now a term on the right arises from each expression for n as a sum of distinct positive integers; its value is $(-1)^k$, where k is the number of terms in the sum. Thus, $c(n)$ is equal to the number of expressions for n as the sum of an even number of distinct parts, minus the number of expressions for n as the sum of an odd number of distinct parts.

This number is evaluated by *Euler's pentagonal numbers formula*:

Proposition 2

$$c(n) = \begin{cases} (-1)^k & \textit{if } n = k(3k-1)/2 \textit{ for some } k \in \mathbb{Z}, \\ 0 & \textit{otherwise.} \end{cases}$$

Putting all this together, the recurrence relation for $p(n)$ is

$$\begin{aligned} p(n) &= \sum_{k\neq 0} (-1)^{k-1} p(n-k(3k-1)/2) \\ &= p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+\cdots \end{aligned}$$

where the summation is over all values of k for which $n-k(3k-1)/2$ is non-negative.

The number of terms in the recurrence grows with n, but only as $O(\sqrt{n})$. So evaluating $p(n)$ for $n \leq N$ requires only $O(n^{3/2})$ additions and subtractions.

Other recurrence relations

There is no recipe for solving more general recrrence relations. We do a few examples for illustration.

Example: derangements Let $d(n)$ be the number of derangements of $\{1,\dots,n\}$ (permutations which have no fixed points). We obtain a recurrence relation as follows. Each derangement maps n to some i with $1 \leq i \leq n-1$, and by symmetry each i occurs equally often. So we need only count the derangements mapping n to $n-1$, and multiply by $n-1$.

We divide these derangements into two classes. The first type map $n-1$ back to n. Such a permutation must be a derangement of $\{1,\dots,n-2\}$ composed with the transposition $(n-1,n)$; so there are $d(n-2)$ such. The second type map i to n for some $i \neq n-1$. Replacing the sequence $i \mapsto n \mapsto n-1$ by the sequence $i \mapsto n-1$, we obtain a derangement of $n-1$; every such derangement arises. So there are $d(n-1)$ deraggements of this type.

Thus,

$$d(n) = (n-1)(d(n-1)+d(n-2)).$$

There is a simpler recurrence satisfied by $d(n)$, which can be deduced from this one, namely

$$d(n) = nd(n-1)+(-1)^n.$$

To prove this by induction, suppose that it is true for $n-1$. Then $(n-1)d(n-2) = d(n-1)-(-1)^{n-1}$; so $d(n) = (n-1)d(n-1)+d(n-1)+(-1)^n$, and the inductive step is proved. (Starting the induction is an exercise.)

Now this is a special case of a general recursion which can be solved, namely

$$x_0 = c, \qquad x_n = p_n x_{n-1} + q_n \text{ for } n \geq 1.$$

We can include the initial condition in the recursion by setting $q_0 = c$ and adoopting the convention that $x_{-1} = 0$.

If $q_n = 0$ for $n \geq 1$, then the solution is simply $x_n = P_n$ for all n, where

$$P_n = c\prod_{i=1}^{n} p_i.$$

So we compare x_n to p_n. Putting $y_n = x_n/P_n$, the recurrence becomes

$$y_0 = 1, \qquad y_n = y_{n-1} + \frac{q_n}{P_n} \text{ for } n \geq 1,$$

with solution

$$y_n = \sum_{i=0}^{n} \frac{q_i}{P_i}.$$

(Remember that $q_0 = P_0 = c$.) Finally,

$$x_n = P_n \sum_{i=0}^{n} \frac{q_i}{P_i}.$$

For derangements, we have $p_n = n$, $c = 1$ (so that $P_n = n!$), and $q_n = (-1)^n$. Thus

$$d(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

It follows that $d(n)$ is the nearest integer to $n!/\mathrm{e}$, since

$$n!/\mathrm{e} - d(n) = n! \sum_{i \geq n+1} \frac{(-1)^i}{i!},$$

and the modulus of the alternating sum of decreasing terms on the right is smaller than that of the first term, which is $n!/(n+1)! = 1/(n+1)$.

Example: Catalan numbers It is sometimes possible to use a recurrence relation to derive an algebraic or differential equation for a generating function for the sequence. If we are lucky, this equation can be solved, and the resulting function used to find the terms in the sequence.

The nth *Catalan number* C_n is the number of ways of bracketing a product of n terms, where we are not allowed to assume that the operation is associatuve or commutative. For example, for $n = 4$, there are five bracketings

$$(a(b(cd))), (a((bc)d)), ((ab)(cd)), ((a(bc))d), (((ab)c)d),$$

so $C_4 = 5$.

Any bracketed product of n terms is of the form (AB), where A and B are bracketed products of i and $n-i$ terms respectively. So

$$C_n = \sum_{i=1}^{n-1} C_i C_{n-i} \text{ for } n \geq 2.$$

Putting $F(x) = \sum_{n\geq 1} C_n x^n$, the recurrence relation shows that F and F^2 agree in all coefficients except $n = 1$. Since $C_1 = 1$ we have $F = F^2 + x$, or $F^2 - F + x = 0$. Solving this equation gives

$$F(x) = \tfrac{1}{2}(1 \pm \sqrt{1-4x}).$$

Since $C_0 = 0$ by definition, we must take the negative sign here.

This expression gives us a rough estimate for C_n: the nearest singularity to the origin is a branchpoint at $1/4$, so C_n grows "like" 4^n. However, we can get the solution explicitly.

From the binomial theorem, we have

$$F(x) = \tfrac{1}{2}\left(1 - \sum_{n\geq 0} \binom{1/2}{n}(-4)^n\right).$$

Hence

$$\begin{aligned}
C_n &= -\frac{1}{2}\binom{1/2}{n}(-4)^n \\
&= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-3}{2}\cdot\frac{2^{2n}}{n!} \\
&= \frac{1}{2^{n+1}}\cdot\frac{(2n-2)!}{2^{n-1}(n-1)!}\cdot\frac{2^{2n}}{n\cdot(n-1)!} \\
&= \frac{1}{n}\binom{2n-2}{n-1}.
\end{aligned}$$

Sometimes we cannot get an explicit solution, but can obtain some information about the growth rate of the sequence.

Example: Wedderburn–Etherington numbers Another interpretation of the Catalan number C_n is the number of rooted binary trees with n leaves, where "left" and "right" are distinguished. If we do not distinguish left and right, we obtain the *Wedderburn–Etherington numbers* W_n.

Such a tree is determined by the choice of trees with i and $n-i$ leaves, but the order of the choice is unimportant. Thus, if $i = n/2$, the number of trees is only $W_i(W_i+1)/2$, rather than W_i^2. For $i \neq n/2$, we simply halve the number. This gives the recurrence

$$W_n = \begin{cases} \frac{1}{2}\sum_{i=1}^{n-1} W_i W_{n-i} & \text{if } n \text{ is odd,} \\ \frac{1}{2}\left(\sum_{i=1}^{n-1} W_i W_{n-i} + W_{n/2}\right) & \text{if } n \text{ is even.} \end{cases}$$

Thus, $F(x) = \sum W_n x^n$ satisfies

$$F(x) = x + \tfrac{1}{2}(F(x)^2 + F(x^2)).$$

This cannot be solved explicitly. We will obtain a rough estimate for the rate of growth. Later, we find more precise asymptotics.

We seek the nearest singularity to the origin. Since all coefficients are real and positive, this will be on the positive real axis. (If a power series with positive real coefficiets converges at $z = r$, then it converges absolutely at any z with $|z| = r$.) Let s be the required point. Then $s < 1$, so $s^2 < s$; so $F(z^2)$ is analytic at $z = s$. Now write the equation as

$$F(z)^2 - 2F(z) + (F(z^2) + 2z) = 0,$$

with "solution"

$$F(z) = 1 - \sqrt{1 - 2z - F(z^2)}$$

(taking the negative sign as before). Thus, s is the real positive solution of

$$F(s^2) = 1 - 2s.$$

Solving this equation numerically (using the fact that $F(s^2)$ is the sum of a convergent Taylor series and can be estimated from knowledge of a finite number of terms), we find that $s \approx 0.403\ldots$, so that W_n grows "like" $(2.483\ldots)^n$.

We will find more precise asymptotics for W_n later in the course.

Example: Bell numbers We already calculated the exponential generating function for the Bell numbers. Here is how to do it using the recurrence relation

$$B(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k).$$

Multiply by $x^n/n!$ and sum over n: the e.g.f $F(x)$ is given by

$$F(x) = \sum_{n\geq 0} \frac{x^n}{n!} \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k).$$

Differentiating with respect to x we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} F(x) &= \sum_{n\geq 1} \frac{x^{n-1}}{(n-1)!} \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k) \\ &= \sum_{l\geq 0} \frac{x^l}{l!} \sum_{m\geq 0} \frac{B(m)x^m}{m!}. \end{aligned}$$

Here we use new variables $l = k-1$ and $m = n-k$; the constraints of the original sum mean that l and m independently take all natural number values. Hence

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) = \exp(x) F(x).$$

This first-order differential equation can be solved in the usual way with the initial condition $F(0) = 1$ to give

$$F(x) = \exp(\exp(x) - 1),$$

in agreement with our earlier result.