Queen Mary
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C50 Enumerative & Asymptotic Combinatorics

Notes 10 **Spring 2003**

A number of non-trivial analytic results have been proved for the purpose of obtaining asymptotic formulae for combinatorially defined numbers. These include theorems of Hayman, Meir and Moon, and Bender. I will not give proofs of these theorems, but treat them as black boxes and give examples to illustrate their use.

Hayman's Theorem

Hayman's Theorem is an important result on the asymptotic behaviour of the coefficients of certain *entire* functions (i.e., functions which are analytic in the entire complex plane).

The theorem applies only to a special class of such functions, the so-called *H-admissible* or *Hayman-admissible* functions. Rather than attempt to give a general definition of this class, I will state a theorem of Hayman showing that it is closed under certain operations, which suffice to show that any function in which we are interested is H-admissible. See Hayman's paper in the bibliography, or Odlyzko's survey.

Theorem 1 *(a) If f is H-admissible and p is a polynomial with real coefficients, then $f+p$ is H-admissible.*

(b) If p is a non-constant polynomial with real coefficients such that $\exp(p(x)) = \sum q_n x^n$ with $q_n > 0$ for $n \geq n_0$, then $\exp(p(x))$ is H-admissible.

(c) If p is a non-constant real polynomial with leading term positive, and f is H-admissible, then $p(f(x))$ is H-admissible.

(d) If f and g are H-admissible, then $\exp(f(x)$ and $f(x)g(x)$ are H-admissible.

Corollary 2 *The exponential function is H-admissible.*

Now *Hayman's Theorem* is the following.

Theorem 3 *Let $f(x) = \sum_{n\geq 0} f_n x^n$ be H-admissible. Let $a(x) = xf'(x)/f(x)$ and $b(x) = xa'(x)$, and let r_n be the smallest positive root of the equation $a(x) = n$. Then*

$$f_n \sim \frac{1}{\sqrt{2\pi b_n}} f(r_n) r_n^{-n}.$$

Example: Stirling's formula Take $f(x) = \exp(x)$ (we have noted that this function is admissible), so that $f_n = 1/n!$. Now $a(x) = x = b(x)$, and $r_n = n$. Thus

$$\frac{1}{n!} = \frac{1}{\sqrt{2\pi n}} \mathrm{e}^n n^{-n},$$

which is just Stirling's formula the other way up!

Example: Bell numbers Let $f(x) = \exp(\exp(x) - 1)$, so that $f_n = B(n)/n!$, where $B(n)$ is the number of partitions of an n-set. This function is H-admissible. Now $a(x) = x\mathrm{e}^x$ and $b(x) = (x + x^2)\mathrm{e}^x$.

The number r_n is the smallest positive solution of $x\mathrm{e}^x = n$. In terms of this, we have

$$\frac{B(n)}{n!} \sim \frac{1}{\sqrt{2\pi n(1 + r_n)}} \mathrm{e}^{n/r_n - 1} r_n^{-n},$$

and so by Stirling's formula,

$$B(n) \sim \frac{1}{\sqrt{1 + r_n}} \left(\frac{n}{\mathrm{e} r_n}\right)^n \mathrm{e}^{n/r_n - 1}.$$

Of course, this is not much use without a good estimate for r_n. However, for $n = 100$, the right-hand side is within 0.4% of $B(100)$.

In fact, it can be shown that

$$r_n = \log n - \log\log n + O\left(\frac{\log\log n}{\log n}\right),$$

from which it can be deduced that

$$\log B(n) \sim n\log n - n\log\log n - n.$$

The theorem of Meir and Moon

The theorem of Meir and Moon (which has been generalised by Bender) gives the asymptotics of the coefficients of a power series defined by Lagrange inversion (compare Notes 8). Typically we have to find the inverse function of f. Setting $\phi(x) = x/f(x)$, the inverse function g is given by the functional equation $g(y) = y\phi(g(y))$. Replacing y by x and g by f, the theorem is as follows.

Theorem 4 *Let $y = f(x) = \sum f_n x^n$ satisfy the equation*

$$y = x\Phi(y),$$

where Φ is analytic in some neighbourhood of the origin, with $\Phi(x) = \sum a_n x^n$. Suppose that the following conditions hold:

(a) $a_0 = 1$ *and* $a_n \geq 0$ *for* $n \geq 0$.

(b) $\gcd\{n : a_n > 0\} = 1$.

(c) There is a positive real number α, inside the circle of convergence of Φ, satisfying

$$\alpha\Phi'(\alpha) = \Phi(\alpha).$$

Then

$$f_n \sim Cn^{-3/2}\beta^n,$$

where $C = \sqrt{\Phi(\alpha)/2\pi\Phi''(\alpha)}$ and $\beta = \Phi(\alpha)/\alpha = \Phi'(\alpha)$.

Example: Rooted trees The generating function $y = T^*(x)$ for labelled rooted trees satisfies

$$y = x\exp(y).$$

The exponential function converges everywhere, and the solution of $\alpha\exp(\alpha) = \exp(\alpha)$ is clearly $\alpha = 1$, so that $\beta = \mathrm{e}$ and $C = \sqrt{1/2\pi}$. Hence the number T_n^* of labelled rooted trees on n vertices satisfies

$$\frac{T_n^*}{n!} = \frac{1}{\sqrt{2\pi}} n^{-3/2} \mathrm{e}^n.$$

Since $T_n^* = n^{n-1}$ by Cayley's Theorem, we obtain

$$n! \sim \sqrt{2\pi} \frac{n^{n+1/2}}{\mathrm{e}^n},$$

in other words, Stirling's formula.

Bender's Theorem

Bender's Theorem generalises the theorem of Meir and Moon by treating a very much more general class of implicitly defined functions. Thus, y will be defined as a function of x by the equation $F(x,y) = 0$. In the case of Meir and Moon, we have $F(x,y) = y - x\Phi(y)$.

Theorem 5 *Suppose that* $y = f(x)$ *is defined implicitly by the equation* $F(x,y) = 0$*, and let* $f(x) = \sum_{n\geq 0} f_n x^n$*. Suppose that there exist real numbers* ξ *and* η *such that*

(a) F *is analytic in a neighbourhood of* (ξ,η)*;*

(b) $F(\xi,\eta) = 0$ *and* $F_y(\xi,\eta) = 0$*, but* $F_x(\xi,\eta) \neq 0$ *and* $F_{yy}(\xi,\eta) \neq 0$ *(subscripts denote partial derivatives);*

(c) the only solution of $F(x,y) = F_y(x,y) = 0$ *with* $|x| \leq \xi$ *and* $|y| \leq \eta$ *is* $(x,y) = (\xi,\eta)$*.*

Then

$$f_n \sim Cn^{-3/2}\xi^{-n},$$

where

$$C = \sqrt{\frac{\xi F_x(\xi,\eta)}{2\pi F_{yy}(\xi,\eta)}}.$$

Note that the condition $F_y(x,y) \neq 0$ is required for the Implicit Function Theorem. So we expect that (ξ,η) will be the nearest point to the origin at which the function $f(x)$ defined in this way has a singularity, so that its radius of convergence is ξ. Bender's theorem is a precise statement about its asymptotics which is much stronger than merely saying that $\lim_{n\to\infty}(f_n)^{1/n} = \xi^{-1}$.

Example: Wedderburn–Etherington numbers Recall from Chapter 3 that the generating function for these numbers satisfies

$$f(x) = x + \frac{1}{2}(f(x)^2 + f(x^2)).$$

Here we have $F(x,y) = y - x - (y^2 + g(x))/2$, where $g(x) = f(x^2)$, which we regard as a "known" function (using a truncation of its Taylor series to approximate it).

The equation $F_y(\xi,\eta) = 0$ gives us that $\eta = 1$; the the equation $F(\xi,\eta) = 0$ then gives $g(\xi) = 1 - 2\xi$. This equation can be solved numerically (it is the same one we solved in Chapter 3 to find the radius of convergence of $f(x)$). The remaining conditions of the theorem can then be verified.

We obtain $\xi^{-1} = 2.483\ldots$, and hence

$$f_n \sim Cn^{-3/2}\xi^{-n},$$

where C can also be found numerically if desired.