C50 Enumerative & Asymptotic Combinatorics

Exercises 8 **Spring 2003**

---

1 Let k be fixed and $n \to \infty$. Prove that

$$S(n,k) \sim k^n/k! \qquad \text{and} \qquad S(n,n-k) \sim n^{2k}/2^k k!\,.$$

Find similar asymptotic estimates for $s(n,k)$ and $s(n,n-k)$.

2 Prove that the denominator of the nth Bernoulli number B_n divides $(n+1)!$. (In other words, prove that $(n+1)!\,B_n$ is an integer.)

3 Prove that, for $k > 0$,

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1}\left(B_{k+1}(n+1) - B_{k+1}(0)\right),$$

where $B_{k+1}(t)$ is the Bernoulli polynomial of degree $k+1$. Hence derive Faulhaber's formula.

4 Apply the Euler–Maclaurin sum formula to the function $f(x) = 1/x$ to show

$$\sum_{i=1}^{n} \frac{1}{i} \sim \log n + \gamma - \sum \frac{B_k}{kn^k}.$$

where γ is *Euler's constant*. (Its value is approximately $0.5772157\ldots$.)

5 Let s_n be the number of permutations of $\{1,\ldots,n\}$ which are equal to their inverses. Prove that

$$\sum_{n\geq 0} \frac{s_n x^n}{n!} = \exp\left(x + \frac{x^2}{2}\right),$$

and use Hayman's Theorem to show that

$$s_n \sim \frac{1}{\sqrt{2}}\left(\frac{n}{\mathrm{e}}\right)^{n/2} \mathrm{e}^{\sqrt{n}-1/4}.$$