Queen Mary
University of London

C50 Enumerative & Asymptotic Combinatorics

Exercises 7 — Spring 2003

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1 Count the labelled trees in which the vertex i has valency a_i for $1 \le i \le n$, where $a_1, \ldots, a_n$ are positive integers with sum $2n-2$.

2 Show that the cycle index for the species $\mathcal{C}$ of circular structures is

$$Z(\mathcal{C}) = 1 - \sum_{m \ge 1} \frac{\phi(m)}{m} \log(1 - s_m).$$

Use the fact that

$$\mathcal{P} \sim \mathcal{S}[\mathcal{C}]$$

to show that

$$Z(\mathcal{P}) = \prod_{n \ge 1} (1 - s_n)^{-1}.$$

Can you give a direct proof of this?

3 Use the result of the preceding exercise, and the fact that $c_n = 1$ for all n (where c_n is the number of unlabelled n-element structures in $\mathcal{C}$) to prove the identity

$$\prod_{m \ge 1} (1 - x^m)^{-\phi(m)/m} = \exp(x/(1-x)).$$

4 Suppose that g_n is the number of unlabelled n-element objects in the species $\mathcal{G}$. Show that the generating function for unlabelled structures in $\mathcal{S}[\mathcal{G}]$ is

$$\prod_{n \ge 1} (1 - x^n)^{-g_n}.$$

Verify this combinatorially in the case $\mathcal{G} = \mathcal{S}$. How would you describe the objects of $\mathcal{S}[\mathcal{S}]$?

5 Let $\mathcal{G}$ be a species. The *Stirling numbers* of $\mathcal{G}$ are the numbers $S(\mathcal{G})(n,k)$, defined to be the number of partitions of an n-set into k parts with a $\mathcal{G}$-object on each part.

(a) Prove that, for $\mathcal{G} = \mathcal{S}$, $\mathcal{C}$ and $\mathcal{L}$ respectively, the Stirling numbers are respectively the Stirling numbers $S(n,k)$ of the second kind, the unsigned Stirling numbers $|s(n,k)|$ of the first kind, and the Lah numbers $L(n,k)$ respectively.

(b) Let $S(\mathcal{G})$ be the lower triangular matrix of Stirling numbers of $\mathcal{G}$. Prove that

$$S(\mathcal{G})S(\mathcal{H}) = S(\mathcal{H}[\mathcal{G}]).$$

(c) Let (a_n) and (b_n) be sequences of positive integers with exponential generating functions $A(x)$ and $B(x)$ respectively. Prove that the following two conditions are equivalent:

- $a_0 = b_0$ and $b_n = \sum_{k=1}^{n} S(\mathcal{G})(n,k)a_k$ for $n \geq 1$;
- $B(x) = A(G(x) - 1)$.

6 A *forest* is a graph whose connected components are trees. Show that there is a bijection between labelled forests of rooted trees on n vertices, and labelled rooted trees on $n+1$ vertices with root $n+1$.

Hence show that, if a forest of rooted trees on n vertices is chosen at random, then the probability that it is connected tends to the limit $1/\mathrm{e}$ as $n \to \infty$.

Remark It is true but harder to prove that the analogous limit for unrooted trees is $1/\sqrt{\mathrm{e}}$.

7 Let $\mathcal{U}$ be the *subset* species: a $\mathcal{U}$-object consists of a distinguished subset of its ground set. Calculate the cycle index of $\mathcal{U}$. Hence or otherwise prove that the enumeration functions of $\mathcal{U}$ are

$$\begin{aligned} U(x) &= \exp(2x), \\ u(x) &= (1-x)^{-2}. \end{aligned}$$