Queen Mary
University of London

C50 Enumerative & Asymptotic Combinatorics

Exercises 6 **Spring 2003**

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1 Let $A_1, \dots, A_n$ be subsets of X. For $J \subseteq N = \{1, \dots, n\}$, let A_J consist of the points of X lying in A_i for all $i \in J$, and B_J the points lying in A_i if $i \in J$ and not if $i \notin J$. Show that

$$|B_J| = \sum_{J \subseteq K \subseteq N} (-1)^{|K \setminus J|} |A_K|.$$

2 Let real numbers a_J and b_J be given for each subset J of $N = \{1, \dots, n\}$. Prove that the following conditions are equivalent:

(a) $a_J = \sum_{I \subseteq J} b_I$ for all $J \subseteq N$;

(b) $b_J = \sum_{I \subseteq J} (-1)^{|J \setminus I|} a_I$ for all $J \subseteq N$.

3 By taking the numbers a_J and b_J of the preceding exercise to depend only on the cardinality j of J, show that the following statements are equivalent for two sequences (x_i) and (y_i):

(a) $x_j = \sum_{i=0}^{j} y_i$;

(b) $y_j = \sum_{i=0}^{j} (-1)^{j-i} y_i$.

4 Prove that

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

5 Let x and y be elements of a poset X, with $x \leq y$. A *chain* from x to y is a sequence $x = x_0, x_1, \ldots, x_l = y$ with $x_{i-1} < x_i$ for $i = 1, \ldots, l$; its *length* is l. Show that

$$\mu(x,y) = \sum_{c \in C} (-1)^{l(c)},$$

where C is the set of all chains from x to y, and $l(c)$ is the length of c.

6 Let $d(n)$ be the number of divisors of the positive integer n. Prove that

$$\sum_{m|n} d(m)\mu(n/m) = 1$$

for $n > 1$.

7 Let $\mathcal{P}(X)$ denote the poset whose elements are the partitions of the set X, with $P \leq Q$ if P refines Q (that is, every part of P is contained in a part of Q). Let E be the partition into sets of size 1. Suppose that $|X| = n$.

(a) Show that, if the parts of P have sizes $a_1, \ldots, a_r$, then

$$\mu(E,P) = (-1)^{n-r}(a_1 - 1)! \cdots (a_r - 1)!\,.$$

(b) Show that any interval $[P,Q]$ is isomorphic to a product of posets of the form $\mathcal{P}(X_j)$, and hence calculate $\mu(P,Q)$.

8 Let G be the cyclic group consisting of all powers of the permutation

$$g : 1 \to 2 \to \cdots \to n \to 1.$$

Show that the cycle index of G is

$$Z(G) = \frac{1}{n} \sum_{m|n} \phi(n/m) s_{n/m}^{m},$$

where ϕ is Euler's function.

9 A necklace is made of n beads of q different colours. Necklaces which differ only by a rotation are regarded as the same. Show that the number of different necklaces is

$$\frac{1}{n} \sum_{m|n} q^m \phi(n/m),$$

while the number which have no rotational symmetry is

$$\frac{1}{n}\sum_{m|n} q^m \mu(n/m).$$

(Notice that, if q is a prime power, the second expression is equal to the number of monic irreducible polynomials of degree n over $\mathrm{GF}(q)$. Finding a bijective proof of this fact is much harder!)

10 A function F on the natural numbers is said to be *multiplicative* if

$$\gcd(m,n) = 1 \Rightarrow F(mn) = F(m)F(n).$$

(a) Suppose that F and G are multiplicative. Show that the function H defined by

$$H(n) = \sum_{k|n} F(k)G(n/k)$$

is multiplicative.

(b) Show that the Möbius and Euler functions are multiplicative.

(c) Let $d(n)$ be the number of divisors of n, and $\sigma(n)$ the sum of the divisors of n. Show that d and σ are multiplicative.

11 Prove the following "approximate version" of PIE:

Let $A_1,\ldots,A_n,A'_1,\ldots,A'_n$ be subsets of a set X. For $I \subseteq N = \{1,\ldots,n\}$, let

$$a_I = \left|\bigcap_{i\in I} A_i\right|, \qquad a'_I = \left|\bigcap_{i\in I} A'_i\right|.$$

If $a_I = a'_I$ for all *proper* subsets I of N, then $|a_N - a'_N| \leq |X|/2^{n-1}$.