Queen Mary
University of London

C50 Enumerative & Asymptotic Combinatorics

Exercises 5 Spring 2003

---

1 Let G be a permutation group on a finite set X, where $|X| = n > 1$. Suppose that G is *transitive* on X (this means that G has only one orbit on X). Prove that there is an element of G which is a derangement of X (that is, which has no fixed point). Show further that at least a fraction $1/n$ of the elements of G are derangements.

2 Use the Cycle Index Theorem to write down a polynomial in two variables x and y in which the coefficient of $x^i y^j$ is the number of cubes in which the faces are coloured red, white and blue, having i red and j blue faces, up to rotations of the cube.

3 Find a formula for the number of ways of colouring the faces of the cube with r colours, up to rotations of the cube. Repeat this exercise for the other four Platonic solids.

4 A necklace has ten beads, each of which is either black or white, arranged on a loop of string. A cyclic permutation of the beads counts as the same necklace. How many necklaces are there? How many are there if the necklace obtained by turning over the given one is regarded as the same?

5 Two actions of a group G, on sets X and Y, are said to be *isomorphic* if there is a bijection $\theta : X \to Y$ such that $\theta(x)^g = \theta(x^g)$ for all $x \in X$, $g \in G$.

(a) Prove that any transitive action of G is isomorphic to the action on the set of right cosets of a subgroup H of G by right multiplication. Prove that two transitive actions of G on sets X and Y are isomorphic if and only if the stabilisers of points in X and Y are conjugate. (The *stabiliser* of x is the subgroup

$$\{g \in G : x^g = x\}$$

of G.)

(b) Let G be a group with finitely many subgroups of index n for all n. Let $c_n(G)$ be the number of conjugacy classes of subgroups of index n in G, and $d_n(G)$ the number of G-spaces with n points (up to isomorphism). Prove that

$$\sum_{n=0}^{\infty} d_n(G)t^n = \exp\left(\sum_{k=1}^{\infty} \frac{c(t^k)}{k}\right),$$

where

$$c(t) = \sum_{n=1}^{\infty} c_n t^n.$$