Queen Mary
University of London

C50 Enumerative & Asymptotic Combinatorics

Exercises 4 **Spring 2003**

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1 Prove that, for $0 \le k \le n$,

$$\begin{bmatrix} 2n \\ 2k \end{bmatrix}_{-1} = \begin{bmatrix} 2n+1 \\ 2k \end{bmatrix}_{-1} = \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix}_{-1} = \binom{n}{k},$$

and

$$\begin{bmatrix} 2n+2 \\ 2k+1 \end{bmatrix}_{-1} = 0.$$

(I am grateful to Aidan Bruen for this exercise.)

2 (a) Prove that, for $0 < k < n$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + (q^{n-1}-1)\begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q.$$

(b) Let

$$F_q(n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

so that, if q is a prime power, then $F_q(n)$ is the total number of subspaces of an n-dimensional vector space over $\mathrm{GF}(q)$. Prove that

$$F_q(0) = 1, F_q(1) = 2, \quad F_q(n) = 2F_q(n-1) + (q^{n-1}-1)F_q(n-2) \text{ for } n \ge 2.$$

(c) Deduce that, if $q > 1$, then $F_q(n) \ge c\, q^{n^2/4}$ for some constant c (depending on q).

3 This exercise shows that the Gaussian coefficients have a counting interpretation for all positive integer values of q (not just prime powers).

Suppose that q is an integer greater than 1. Let Q be a finite set of cardinality q containing two distinguished elements 0 and 1. We say that a $k \times n$ matrix with entries from Q is in *reduced echelon form* if the following conditions hold:

- If a row has any non-zero entries, then the first such entry is 1 (such entries are called “leading 1”);
- if $i < j$ and row j is non-zero, then row i is also non-zero, and its leading 1 occurs to the left of the leading 1 in row j;
- if a column contains the leading 1 of some row, then all other entries in that column are 0.

Prove that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of $k \times n$ matrices in reduced echelon form with no rows of zeros.

4 A matrix is said to be in *echelon form* if it satisfies the first two conditions in the definition of reduced echelon form. Show that, if q is an integer greater than 2, the right-hand side of the q-binomial theorem with $x = 1$ counts the number of $n \times n$ matrices in echelon form.

How many $n \times n$ matrices in reduced echelon form are there?

5 Let $h_k(x_1, \ldots, x_n)$ be the *complete symmetric function* of degree k in the indeterminates $x_1, \ldots, x_n$ (the sum of *all* monomials of degree k that can be formed using these indeterminates). For example,

$$h_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1.$$

Prove that

(a) $h_k(1, 1, \ldots, 1) = \binom{n+k-1}{k}$;

(b) $h_k(1, q, \ldots, q^{n-1}) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$ for $q \neq 1$.

6 The second proof in the notes of the formula for the number of monic irreducible polynomials over $\mathrm{GF}(q)$ shows that the number is exactly what is required if every element of $\mathrm{GF}(q^n)$ is the root of a unique monic irreducible polynomial of degree dividing n over $\mathrm{GF}(q)$. Turn the argument around to gove a counting proof of the existence and uniqueness of $\mathrm{GF}(q^n)$, given that of $\mathrm{GF}(q)$. Deduce the existence and uniqueness of $\mathrm{GF}(q)$ for all prime powers q.