Queen Mary
University of London

C50 Enumerative & Asymptotic Combinatorics

Exercises 3 **Spring 2003**

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1 Some questions on Fibonacci numbers.

(a) Show that the number of expressions for n as an ordered sum of ones and twos is F_n.

(b) Verify the following formula for the sloping diagonals of Pascal's triangle:

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} = F_n.$$

(c) Let n be a positive integer. Write down all expressions for n as an ordered sum of positive integers. For each such expression, multiply the summands together; then add the resulting products. Prove that the answer is F_{2n-1}.

(d) In (c), if instead of multiplying the summands, we multiply 2^{d-2} for each summand $d > 2$, then the answer is F_{2n-2}.

(e) Prove that, for $n \geq 0$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n+2} = \begin{pmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{pmatrix}.$$

(f) Use (e) to show that F_n can be computed with $O(\log n)$ arithmetic operations on integers.

2 Let A be a finite set of positive integers. Suppose that the currency of a certain country has A as the set of denominations. Prove that the number $f(n)$ of ways of paying a bill of n units, where coins are paid in order, has generating function $1/(1 - \sum_{a \in A} x^a)$.

Suppose that $A = \{1, 2, 5, 10\}$. Prove that $f(n) \sim c\,\alpha^n$ for some constants c and α, and estimate α.

3 Let a be a binary string of length k with correlation polynomial $C_a(x)$. A random binary sequence is obtained by tossing a fair coin, recording 1 for heads and 0 for tails. Let E_a be the expected number of coin tosses until the first occurrence of a as a consecutive substring. Prove that E_a is the sum, over n, of the probability that a doesn't occur in the first n terms of the sequence. Deduce that $E_a = 2^k\, C_a(1/2)$.

4 This exercise is due to Wilf, and illustrates his "snake oil" method.

(a) Prove that

$$\sum_{n\geq 0} \binom{n+k}{2k} x^{n+k} = \frac{x^{2k}}{(1-x)^{2k+1}}.$$

(b) Let

$$a_n = \sum_{k=0}^{n} \binom{n+k}{2k} 2^{n-k}$$

for $n \geq 0$. Prove that the ordinary generating function for (a_n) is

$$\sum_{n\geq 0} a_n x^n = \frac{1-2x}{(1-x)(1-4x)},$$

and deduce that $a_n = (2^{2n+1}+1)/3$ for $n \geq 0$.

(c) Write down a linear recurrence relation with constant coefficients satisfied by the numbers a_n.

5 Let s_n be the number of partitions of an n-set into parts of size 1 or 2 (equivalently, the number of permutations of an n-set whose square is the identity). Show that

$$s_n = s_{n-1} + (n-1)s_{n-2} \text{ for } n \geq 2,$$

and hence find the exponential generating function for (s_n) in closed form.

6 Let a_n be the number of strings that can be formed from n distinct letters (using each letter at most once, and including the empty string). Prove that

$$a_0 = 1, \qquad a_n = na_{n-1} + 1 \text{ for } n \geq 1,$$

and deduce that $a_n = \lfloor \mathrm{e}\, n! \rfloor$. What is the exponential generating function for this sequence?