Queen Mary
University of London

# C50 Enumerative & Asymptotic Combinatorics

**Exercises 2** **Spring 2003**

---

**1** Show that the number of ways of selecting $k$ objects from a set of $n$ distinguished objects, if we allow the same object to be chosen more than once and pay no attention to the order in which the choices are made, is $\binom{n+k-1}{k}$.

**2** Prove that, if $n$ is even, then

$$\frac{2^n}{n+1} \le \binom{n}{n/2} \le 2^n.$$

Use Stirling's formula to prove that

$$\binom{n}{n/2} \sim \frac{2^n}{\sqrt{\pi n/2}}.$$

How accurate is this estimate for small $n$?

**3** Use the method of the preceding exercise, together with the Central Limit Theorem, to deduce the constant in Stirling's formula.

**4** Prove directly that, if $0 \le k < n$, then

$$\sum_m (-1)^{m-k} \binom{n}{m} \binom{m}{k} = \sum_m (-1)^{n-m} \binom{n}{m} \binom{m}{k} = 0.$$

**5** Formulate and prove an analogue of Proposition 9 in the notes for binomial coefficients.

**6** Let $B(n)$ be the number of partitions of $\{1, \ldots, n\}$. Prove that

$$\sqrt{n!} \le B(n) \le n!\,.$$

**7** Prove that $\log n!$ is greater than $n \log n - n + 1$ and differs from it by at most $\frac{1}{2}\log n$. Deduce that

$$\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1/2}}{e^{n-1}}.$$

**8** Let $c(n)$ be the number of connected permutations on $\{1,\ldots,n\}$. (A permutation $\pi$ is *connected* if there does not exist $k$ with $1 \leq k \leq n-1$ such that $\pi$ maps $\{1,\ldots,k\}$ to itself.) Prove that

$$n! = \sum_{k=1}^{n} c(k)(n-k)!\,,$$

and deduce that

$$\left(1 + \sum_{n\geq 1} n!\, x^n\right)^{-1} = 1 - \sum_{n\geq 1} c(n)x^n,$$

**9** Prove that

$$(-1)^{n-k}\binom{n}{k} = \binom{-n+k-1}{k}$$

for $0 \leq k \leq n$. Use this and Proposition 3 to prove the Binomial Theorem for negative integer exponents.

**10** Prove that

$$\sum_{n\geq k} \frac{s(n,k)x^n}{n!} = \frac{(\log(1+x))^k}{k!}$$

for $k \geq 1$. What happens when this equation is summed over $k$?

**11** What is the relation between the numbers $T(n,k)$ defined in Exercise 3 on Sheet 1 and Stirling numbers?

**12** A *total preorder* on a set $X$ is a binary relation $\rho$ on $x$ which is symmetric and transitive and satisfies the condition that, for all $x, y \in X$, either $x\,\rho\,y$ or $y\,\rho\,x$ holds.

(a) Let $\rho$ be a total preorder on $X$. Define a relation $\sigma$ on $X$ by the rule that $x\,\sigma\,y$ if and only if both $x\,\rho\,y$ and $y\,\rho\,x$ hold. Prove that $\sigma$ is an equivalence relation whose equivalence classes are totally ordered by $\rho$. Show that $\rho$ is determined by $\sigma$ and the ordering of its equivalence classes. Show further that any equivalence relation and any total ordering of its equivalence classes aise in this way from a total preorder.

(b) Show that the number of total preorders of an $n$-set is

$$\sum_{k=1}^{n} S(n,k)k!\,.$$

(c) Show that the exponential generating function for the sequence in (b) is $1/(2-\exp(x))$.

(d) What can you deduce about the asymptotic behaviour of the sequence?

**13** For $1 \le k \le n$, the *Lah number* $L(n,k)$ is defined by the formula

$$L(n,k) = \sum_{m=k}^{n} |s(n,m)| S(m,k).$$

(That is, the Lah numbers form a lower triangular matrix which is the product of the matrices of unsigned Stirling numbers of the first and second kinds. They are sometimes called Stirling numbers of the third kind.) Prove that

$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}.$$

**14** Prove that

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k}^2 &= \binom{2n}{n}; \\ \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 &= \begin{cases} 0 & \text{if } n \text{ is odd;} \\ (-1)^{n/2}\binom{n}{n/2} & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

**15** Prove that the generating function for the central binomial coefficients is

$$\sum_{n\ge 0} \binom{2n}{n} x^n = (1-4x)^{-1/2},$$

and deduce that

$$\sum_{k=1}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n.$$

[Note: Finding a counting proof of this identity is quite challenging!]