Queen Mary
University of London

C50 Enumerative & Asymptotic Combinatorics

Exercises 1 — Spring 2003

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1 Prove directly that $(1-x)^{-1} = \sum_{n\geq 0} x^n$ (in the ring of formal power series).

2 Suppose that a collection of complex power series all define functions analytic in some neighbourhood of the origin, and satisfy some identity there. Are we allowed to conclude that this identity holds between the series regarded as formal power series?

3 Show that the identity $\exp(\log(1+x)) = 1+x$ between formal power series is equivalent to the equation

$$\sum_{k=1}^{n} \frac{(-1)^k}{k!} T(n,k) = 0,$$

for $n > 1$, where $T(n,k)$ is computed as follows: write n as an ordered sum of k positive integers $a_1, \ldots, a_k$ in all possible ways; for each such expression compute the product $a_1 \cdots a_k$; and sum the reciprocals of the resulting numbers.

What is the analogous interpretation of the identity $\log(1+(\exp(x)-1)) = x$?

4 Show that the identity $\exp(x+y) = \exp(x)\exp(y)$ is equivalent to the Binomial Theorem for all positive integer exponents.

5 Prove that $n^k = o(c^n)$ for any constants $k > 0$ and $c > 1$, and that $\log n = o(n^\varepsilon)$ for any $\varepsilon > 0$.

6 Let $f(n)$ be the number of partitions of an n-set into parts of size 2.

(a) Prove that

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ 1\cdot 3\cdot 5\cdots(n-1) & \text{if } n \text{ is even.} \end{cases}$$

(b) Prove that the exponential generating function for the sequence $(f(n))$ is $\exp(x^2)$.

(c) Prove that

$$f(n) \sim \sqrt{2}\left(\frac{2n}{\mathrm{e}}\right)^n$$

for even n.

7 Show that it is possible to generate all subsets of $\{1, \ldots, n\}$ successively in such a way that each subset differs from its predecessor by the addition or removal of precisely one element. (Such a sequence is known as a *Gray code*.)