Queen Mary
University of London

C50 Enumerative & Asymptotic Combinatorics

Solutions to Sample Exam — Spring 2003

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1 (a) The definition is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

The fact that it is a polynomial is most easily seen by induction on n using the recurrence relation

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \text{ for } 0 < k < n.$$

The degree of the polynomial is most easily seen from the original definition: the highest power of q has coefficient 1 and exponent

$$(n+(n-1)+\cdots(n-k+1))-(k+(k-1)+\cdots+1 = k(n-k).$$

(b) By l'Hôpital's rule,

$$\lim_{q\to1}\frac{q^a-1}{q^b-1} = \lim_{q\to1}\frac{aq^{a-1}}{bq^{b-1}} = \frac{a}{b}.$$

So

$$\lim_{q\to1}\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots1} = \binom{n}{k}.$$

(c) The q-binomial theorem states

$$\prod_{i=1}^{n}(1+q^{i-1}z) = \sum_{k=0}^{n} q^{k(k-1)/2}z^k \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

The proof is by induction using the recurrence relation.

(d) If q is a prime power, then $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of k-dimensional subspaces of an n-dimensional vector space over $\mathrm{GF}(q)$. More generally, if q is an integer greater than 1, and Q a set with distinguished elements 0 and 1, then $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of $k \times n$ matrices in reduced echelon form with entries in Q and no zero rows.

2 (a) The values $(p(0),\ldots,p(4))$ are $(1,1,2,3,5)$: the partitions are $\emptyset$; (1); (2) and $(1,1)$; (3), $(2,1)$ and $(1,1,1)$; and (4), $(3,1)$, $(2,2)$, $(2,1,1)$ and $(1,1,1,1)$.

We have

$$\prod_{i\geq 1}(1-x^i)^{-1} = \prod_{i\geq 1}(1+x^i+x^{2i}+\cdots).$$

Terms in x^n come from expressions $n = a_1+2a_2+\cdots$, each such expression contributing 1. But expressions of this form correspond bijectively to partitions of n, the partition corresponding to the above expression having a_1 parts equal to 1, a_2 parts equal to 2, and so on. So the identity is proved.

Let $\prod_{i\geq 1}(1-x^i) = \sum_n \geq 0 q(n)x^n$. An argument like the above shows that q_n is equal to the number of partitions of n into an even number of distinct parts minus the number of partitions into an odd number of distinct parts. Euler's Pentagonal Numbers Theorem then asserts that

$$q(n) = \begin{cases} (-1)^k & \text{if } n = k(3k\pm 1) \text{ for some integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

Now we have

$$\left(\sum_{n\geq 0} p(n)x^n\right)\left(\sum_{n\geq 0} q(n)x^n\right) = 1,$$

so

$$p(n) = -\sum_{i=1}^{n} q(i)p(n-i) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+\cdots$$

(b) Each partition of $\{1,\ldots,n\}$ gives rise to a partition of n consisting of the cardinalities of the parts. So we have to find how many times each part occurs. For $n = 4$, the partitions (4), $(3,1)$, $(2,2)$, $(2,1,1)$ and $(1,1,1,1)$ give rise to respectively 1, 4, 3, 6 and 1 partitions of the set, so $B(4) = 15$. Similarly the values of $B(0),\ldots,B(3)$ are 1, 1, 2, 5.

We have the recurrence relation

$$B(n) = \sum_{k=1}^{n}\binom{n-1}{k-1}B(n-k)$$

since a partition of $\{1,\ldots,n\}$ is obtained by choosing a set containing n (if this set has cardinality k, this can be done in $\binom{n-1}{k-1}$ ways) and then a partition of the remaining points (in $B(n-k)$ ways).

Let $F(x)$ be the e.g.f. of $(B(n))$. Multiplying the recurrence by $x^{n-1}/(n-1)!$ and summing over n, we obtain

$$\begin{aligned} F'(x) &= \left(\sum_k \frac{x^k}{k!}\right)\left(\sum_m \frac{B(m)x^m}{m!}\right) \\ &= \exp(x)F(x). \end{aligned}$$

Solving this differential equation with initial condition $F(0) = 1$ we obtain

$$F(x) = \exp(\exp(x) - 1).$$

An object of the required species is a finite set with a partition.

3 The binomial theorem for arbitrary exponent n states that

$$(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k,$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

This is an identity of formal power series, or of analytic functions for $|X| < 1$.

By the Binomial Theorem,

$$(1-4x)^{-1/2} = \sum_{k\geq 0} \binom{-1/2}{k} (-4x)^k.$$

The coefficient of x^k is

$$\begin{aligned} & \frac{-1}{2}\frac{-3}{2}\cdots\frac{-(2k-1)}{2}\cdot\frac{(-4)^k}{k!} \\ = & \frac{(2k)!}{2^{2k}k!}\frac{2^{2k}}{k!} \\ = & \binom{2k}{k} \end{aligned}$$

as required.

We have

$$(1-4x)^{-1/2}(1-4x)^{-1/2} = (1-4x)^{-1} = \sum_{n\geq 0} 4^n x^n.$$

The term in x^n on the left is

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k}$$

so the identity holds.

4 Let $|X| = n$. Take indeeterminates $s_1, \ldots, s_n$ and put

$$z(g) = \prod_{i=1}^{n} s_i^{c_i(g)},$$

where $c_i(g)$ is the number of i-cycles in the cycle decomposition of the permutation g. Then the cycle index of the group G is

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} z(g).$$

The Orbit-Counting Lemma states that the number of orbits of G on X is equal to

$$\frac{1}{|G|} \sum_{g \in G} c_1(g),$$

where $c_1(g)$ is the number of fixed points of g.

Take a set of figures with non-negative integer weights, with figure-counting series $A(x) = \sum a_n x^n$, where a_n is the number of figures of weight n (assumed finite). Let the number of orbits of G on the set of functions from X to the set of figures having total weight n be b_n, and define the function-counting series $B(x) = \sum b_n x^n$. Then the Cycle Index Theorem states that

$$B(x) = Z(G; s_i \leftarrow A(x^i) \text{ for } i = 1, \ldots, n).$$

The group of rotations of the octahedron has order 24. It contains

- the identity, fixing all eight faces;
- three rotations of order 2 about diagonals, each with four 2-cycles;
- six rotations of order 4 about diagonals, each with two 4-cycles;
- eight rotations of order 3 about lines joining mod-points of the triangular faces, each with two fixed faces and two 3-cycles;
- six rotations of order 2 about lines joining mid-points of edges, each with four 2-cycles.

So the cycle index is

$$Z(G) = \frac{1}{24}(s_1^8 + 9s_2^4 + 6s_4^2 + 8s_1^2 s_3^2.$$

Let the figure 'red' have weight 0 and the figure 'blue' have weight 1. Then the figure-counting series is $1+x$, so by the Cycle Index Theorem, the required generating function is $Z(G; s_i \leftarrow 1 + x^i)$, which is

$$\frac{1}{24}((1+x)^8 + 9(1+x^2)^4 + 6(1+x^4)^2 + 8(1+x)^2(1+x^3)^2.$$

5 A *partially ordered set* is a set X with a relation $\leq$ saatisfying

- $x \leq y$ and $y \leq x$ if and only if $x = y$;
- $x \leq y$ and $y \leq z$ imply $x \leq z$.

Its *zeta-function* is the function on $X \times X$ satisfying $\zeta(x,y) = 1$ if $x \leq y$, $\zeta(x,y) = 0$ otherwise; its *Möbius function* is the inverse of the zeta-function under the multiplication

$$(fg)(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y).$$

Thus the Möbius function satisfies the recurrence

$$\mu(x,x) = 1, \quad \mu(x,z) = -\sum_{x \leq y < z} \mu(x,y),$$

and we only have to prove the same recurrence for the function

$$\lambda(x,y) = \sum_{n \geq 0} (-1)^n C_n(x,y).$$

Now any chain $(x_0, \ldots, x_n)$ with $x_0 = x$ and $x_n = y$ contributes $(-1)^n$ to $\lambda(x,y)$, but also contributes $(-1)^{n-1}$ to $\sum_{x \leq z < y} \lambda(x,z)$ (taking $z = x_{n-1}$); so the expressions are equal. The induction begins because $\lambda(x,x) = 1$ from the trivial chain.

If the three intermediate points are x, y, z, the chains from 0 to 1 are $(0,1)$, $(0,x,1)$, $(0,y,1)$, $(0,z,1)$; so we have

$$\mu(0,1) = (-1) + 1 + 1 + 1 = 2.$$

6 The only permutation of $\{1, \ldots, n\}$ with n cycles is the identity, so $s(n,n) = 1$. Now consider a permutation with k cycles. There are two possibilities:

- n is a fixed point. Then the permutation corresponds to a permutation of $\{1, \ldots, n-1\}$ with $k-1$ cycles.
- n is not fixed, say $i \to n \to j$. Then there is a unique permutation $\ldots i \to j \ldots$ of $\{1, \ldots, n-1\}$ with k cycles. However, any such permutation gives rise to $n-1$ permutations of $\{1, \ldots, n\}$, since the point n could be inserted in any of $n-1$ positions.

So

$$s(n,k) = s(n-1,k-1) - (n01)s(n-1,k)$$

(the minus sign because the sign is changed in the second case).

The value of $s(n,1)$ is easily found from the recurrence relation. Alternatively, there are $(n-1)!$ permutations with a single cycle: we can start the cycle with 1 and insert the remaining points in the $n-1$ remaining positions in $(n-1)!$ ways, So $s(n,1) = (-1)^{n-1}(n-1)!$.

For $n = 1$ the identity is clear, so we use induction.

$$\begin{aligned} \sum_{k=1}^{n} s(n,k)x^k &= \sum_{k=2}^{n} s(n-1,k-1)x^k - (n-1)\sum_{k=1}^{n-1} s(n-1,k)x^k \\ &= x \cdot x(x-1)\cdots(x-n+2) - (n-1) \cdot x(x-1)\cdots(x-n+2) \\ &= x(x-1)\cdots(x-n+2)(x-n+1), \end{aligned}$$

as required.

The inverse of this matrix is the matrix whose (n,k) entry is the Stirling number $S(n,k)$ of the second kind (the number of partitions of an n-set into k parts) for $1 \leq k \leq n$ and zero for $k > n$.

7 Hayman's Theorem states:

> Let $f(x) = \sum_{n \geq 0} f_n x^n$ be H-admissible. Let $a(x) = x f'(x)/f(x)$ and $b(x) = xa'(x)$, and let r_n be the smallest positive root of the equation $a(x) = n$. Then
>
> $$f_n \sim \frac{1}{\sqrt{2\pi b_n}} f(r_n) r_n^{-n}.$$

This depends on the definition of H-admissible function, which is not given explicitly but we note that many functions (including the exponential function) can be shown to be H-admissible.

This is Cayley's Theorem; several proofs appear in the notes.

Let $f(x) = \sum_{n \geq 0} T_n x^n/n!$ be the e.g.f. for labelled rooted trees. From Cayley's Theorem we know that $T_n = n \cdot n^{n-2} = n^{n-1}$. Now the species $\mathcal{T}$ of rooted trees satisfies

$$\mathcal{T} \sim \mathcal{E} \times \mathcal{S}[\mathcal{T}],$$

since a rooted tree can be regarded as a root joined to a set of rooted trees (here $\mathcal{S}$ is the species of sets). Since the e.g.f. for labelled structures in $\mathcal{S}$ is $\sum x^n/n! = \exp(x)$, we have

$$f(x) = x \exp(f(x)).$$

This shouldn't say 'hence' – sorry. Stirling's formula is most easily found by applying Hayman's theorem directly to the exponential function: we have $f(x) = \exp(x)$, $a(x) = x f'(x)/f(x) = x$, $b(x) = xa'(x) = x$, $r_n = n$ (the solution of $x = a(x) = n$). So

$$\frac{1}{n!} \sim \frac{1}{\sqrt{2\pi n}} \mathrm{e}^n n^{-n},$$

which is just Stirling's formula the other way up.