Queen Mary
University of London

M. Sc. Examination 2003

MTHM C50 Enumerative and Asymptotic Combinatorics

Duration: 3 hours

Date and time:

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Question 1 (a) Define the *Gaussian coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_q$, for integers n, k with $0 \leq k \leq n$. Prove that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q and find its degree.

(b) Show that

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

(c) State and prove the *q-Binomial Theorem*.

(d) Give a counting interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ in the case when q is a prime power.

Question 2 (a) The *partition number* $p(n)$ is the number of partitions of the non-negative integer n. Calculate $p(n)$ for $0 \le n \le 4$.

Prove that
$$\sum_{n \ge 0} p(n)x^n = \prod_{i \ge 1} (1 - x^i)^{-1}.$$

Hence find a recurrence relation for the numbers $p(n)$. (You may use Euler's Pentagonal Numbers Theorem without proof provided you state it clearly.)

(b) The *Bell number* $B(n)$ is the number of partitions of the set $\{1, \ldots, n\}$. Calculate $B(n)$ for $0 \le n \le 4$.

Prove that
$$\sum_{n \ge 0} \frac{B(n)x^n}{n!} = \exp(\exp(x) - 1).$$

(c) Describe a species for which $p(n)$ and $B(n)$ respectively enumerate the unlabelled and labelled objects on a set of n points.

Question 3 (a) State without proof the Binomial Theorem for arbitrary exponent.

(b) Prove that the generating function for the central binomial coefficients is
$$\sum_{n \ge 0} \binom{2n}{n} x^n = (1 - 4x)^{-1/2},$$

and deduce that
$$\sum_{k=1}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} = 4^n.$$

Question 4 Let G be a finite group of permutations of a finite set X. Define the *cycle index polynomial* of G, and state the *Orbit-Counting Lemma* and the *Cycle Index Theorem*.

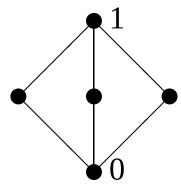
Let G be the group of rotations of a regular octahedron, regarded as a permutation group on the set of eight faces of the octahedron. Calculate the cycle index of G. Hence find a generating function $\sum a_i x^i$ for the number a_i of colourings of the faces with two colours, say red and blue, having exactly i blue faces.

Question 5 What is a *partially ordered set*? What is its *Möbius function*?

A *chain* of length n in a partially ordered set X is a sequence $(x_0, x_1, \dots, x_n)$ with $x_0 < x_1 < \cdots < x_n$. Let $C_n(x,y)$ denote the number of chains of length n with $x_0 = x$ and $x_n = y$. Prove that

$$\mu(x,y) = \sum_{n\geq 0} (-1)^n C_n(x,y).$$

Hence find $\mu(0,1)$ in the poset P shown in the diagram.



Question 6 Let $s(n,k)$ be the Stirling number of the first kind (so that $(-1)^{n-k}s(n,k)$ is the number of permutations of $\{1,\dots,n\}$ with exactly k cycles, for $1 \leq k \leq n$). Prove that $s(n,n) = 1$ and

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k) \text{ for } 1 < k < n.$$

Hence or otherwise prove that $s(n,1) = (-1)^{n-1}(n-1)!$.

Show that

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=1}^{n} s(n,k)x^k.$$

State without proof the inverse of the infinite triangular matrix whose (n,k) entry is $s(n,k)$ for $1 \leq k \leq n$ and is 0 for $k > n$.

Question 7 State *Hayman's Theorem*.

Prove that the number of rooted trees on the set $\{1,\dots,n\}$ is n^{n-1}.

Show that the exponential generating function $f(x)$ for labelled rooted trees satisfies $f(x) = x\exp(f(x))$. Hence deduce *Stirling's asymptotic formula* for $n!$.

 END OF EXAMINATION