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EVERY ERGODIC MEASURE IS UNIQUELY MAXIMIZING

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Abstract. Let $\mathcal{M}_\phi$ denote the set of Borel probability measures invariant under a topological action ϕ on a compact metrizable space X. For a continuous function $f : X \to \mathbb{R}$, a measure $\mu \in \mathcal{M}_\phi$ is called f-maximizing if $\int f\, d\mu = \sup\{\int f\, dm : m \in \mathcal{M}_\phi\}$. It is shown that if μ is any ergodic measure in $\mathcal{M}_\phi$, then there exists a continuous function whose unique maximizing measure is μ. More generally, if $\mathcal{E}$ is a non-empty collection of ergodic measures which is weak* closed as a subset of $\mathcal{M}_\phi$, then there exists a continuous function whose set of maximizing measures is precisely the closed convex hull of $\mathcal{E}$. If moreover ϕ has the property that its entropy map is upper semi-continuous, then there exists a continuous function whose set of equilibrium states is precisely the closed convex hull of $\mathcal{E}$.

1. **Introduction.** Let X be a compact metrizable space, and Γ a topological group or semi-group. Let ϕ be a topological action of Γ on X, i.e., a continuous map $\phi : \Gamma \times X \to X$, $(\gamma, x) \mapsto \phi_\gamma(x)$ such that $\phi_1 = \mathrm{id}_X$ and $\phi_{\gamma'} \circ \phi_\gamma = \phi_{\gamma'\gamma}$ for all $\gamma, \gamma' \in \Gamma$. Let $\mathfrak{B}$ denote the σ-algebra of Borel subsets of X. A Borel probability measure μ on X is called *ϕ-invariant* if $\mu(\phi_\gamma^{-1}A) = \mu(A)$ for all $\gamma \in \Gamma$ and all $A \in \mathfrak{B}$. Let $\mathcal{M}_\phi$ denote the set of ϕ-invariant Borel probability measures. We shall always assume that $\mathcal{M}_\phi$ is non-empty[1]. This is the case whenever Γ is amenable (see e.g. [Gla, p. 97]). In particular, $\mathcal{M}_\phi$ is non-empty when the action ϕ is generated by a single map (in which case $\Gamma = \mathbb{Z}$ or $\mathbb{Z}^+$)[2], or if ϕ is a flow or semi-flow (in which case $\Gamma = \mathbb{R}$ or $\mathbb{R}^+$), by a theorem of Krylov & Bogolioubov [KB] (see [Wal, Cor. 6.9.1]). $\mathcal{M}_\phi$ is convex, and when equipped with the weak* topology it is compact and metrizable.

A measure $\mu \in \mathcal{M}_\phi$ is *ergodic* if $\mu(A)(1 - \mu(A)) = 0$ for every $A \in \mathfrak{B}$ such that $\mu(A \triangle \phi_\gamma^{-1}A) = 0$ for all $\gamma \in \Gamma$. The set of extreme points in $\mathcal{M}_\phi$ is precisely $\mathcal{M}_{\mathrm{erg}}$, the set of ergodic measures in $\mathcal{M}_\phi$.

For a continuous function $f : X \to \mathbb{R}$, a measure $\mu \in \mathcal{M}_\phi$ is called *f-maximizing* if

$$\int f\, d\mu = \sup\left\{\int f\, dm : m \in \mathcal{M}_\phi\right\}.$$

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[1]If $\mathcal{M}_\phi$ is empty then the results in this paper are vacuously true.

[2]$\mathbb{Z}^+$ (respectively $\mathbb{R}^+$) denotes the set of non-negative integers (respectively reals).

The weak* compactness of $\mathcal{M}_\phi$ implies that the set $\mathcal{M}_{\max}(f)$ of f-maximizing measures is non-empty. In general, $\mathcal{M}_{\max}(f)$ is not a singleton, however; for example, if f is a constant, then every $\mu \in \mathcal{M}_\phi$ is f-maximizing.

We shall be interested in *uniquely maximizing measures*, i.e., those measures $\mu \in \mathcal{M}_\phi$ for which there exists a continuous function f such that $\mathcal{M}_{\max}(f) = \{\mu\}$. Any uniquely maximizing measure is necessarily ergodic because, as is readily verified, $\mathcal{M}_{\max}(f)$ is a convex set whose extreme points are the ergodic f-maximizing measures. It turns out that ergodicity is also a *sufficient* condition for an invariant measure to be uniquely maximizing.

Theorem 1. *Let μ be any ergodic invariant Borel probability measure on X. There exists a continuous function $f : X \to \mathbb{R}$ such that μ is the unique f-maximizing measure.*

In fact, Theorem 1 is a consequence of the following, more general result.

Theorem 2. *Let $\mathcal{E}$ be a non-empty subset of $\mathcal{M}_{\mathrm{erg}}$ which is weak* closed in $\mathcal{M}_\phi$. Let $\overline{\mathrm{co}}(\mathcal{E})$ denote its closed convex hull in $\mathcal{M}_\phi$.*

There exists a continuous function $f : X \to \mathbb{R}$ such that the set of f-maximizing measures is precisely $\overline{\mathrm{co}}(\mathcal{E})$.

If $\mathcal{M}_{\mathrm{erg}}$ happens to be a weak* closed subset of $\mathcal{M}_\phi$ (which in general it is not)[3], then the conclusion of Theorem 2 applies if $\mathcal{E}$ is *any* non-empty subset of $\mathcal{M}_{\mathrm{erg}}$.

Theorem 3. *Suppose that $\mathcal{M}_{\mathrm{erg}}$ is a weak* closed subset of $\mathcal{M}_\phi$. For every non-empty subset $\mathcal{E} \subset \mathcal{M}_{\mathrm{erg}}$, there exists a continuous function $f : X \to \mathbb{R}$ such that the set of f-maximizing measures is precisely $\overline{\mathrm{co}}(\mathcal{E})$.*

Indeed, we have the following characterisation of those subsets of $\mathcal{M}_\phi$ which are of the form $\mathcal{M}_{\max}(f)$ for some $f \in C(X)$.

Theorem 4. *The set $\{\mathcal{M}_{\max}(f) : f \in C(X)\}$ is precisely the set of closed faces of $\mathcal{M}_\phi$.*

Most work on maximizing measures has focused on topological actions generated by a single uniformly hyperbolic map T. For such systems, it is known (see [Bou1, Bou2, CG, CLT] or [Jen3, §4]) that if f is sufficiently regular (e.g., Hölder) and has a unique maximizing measure μ, then μ is strictly ergodic (i.e., its topological support $\mathrm{supp}(\mu)$ carries no other invariant measure)[4]. In particular, such a uniquely maximizing measure μ cannot be fully supported. Conversely, every strictly ergodic μ is easily seen to be uniquely maximizing. Indeed, we can *explicitly exhibit* continuous f such that $\mathcal{M}_{\max}(f) = \{\mu\}$; for example, $f(x) = -d(x, \mathrm{supp}(\mu))$ for any metric d which generates the topology on X. On the other hand, from [BJ, Thm. C] it is known that there exist continuous (necessarily non-Hölder) f for which $\mathcal{M}_{\max}(f)$ consists of a single *fully supported* T-invariant measure; in fact, such f form a residual subset of $C(X)$. The proof of [BJ, Thm. C] is non-constructive, however, and begs two questions. Firstly, are there any *obstructions* to an ergodic μ being the unique member of some $\mathcal{M}_{\max}(f)$? Theorem 1 answers this question

[3]For example, if the action is generated by an Axiom A diffeomorphism $T : X \to X$, then $\mathcal{M}_{\mathrm{erg}}$ is a proper dense subset of $\mathcal{M}_\phi$ [Sig].

[4]More usually, a closed invariant set is called *strictly ergodic* if it is both uniquely ergodic and minimal for the action ϕ. There is an obvious one-to-one correspondence between strictly ergodic measures and strictly ergodic sets.

negatively. Secondly, can we exhibit an *explicit* f whose unique maximizing measure is fully supported? This problem remains open, though the knowledge that every ergodic measure is uniquely maximizing may be helpful in resolving it, particularly in cases where *Lebesgue* measure is known to be T-invariant and ergodic. For example, in the case that T is the circle expanding map $T : x \mapsto 2x \pmod 1$, we have the following corollary.

Corollary 1. *Define* $T : x \mapsto 2x \pmod 1$ *on the circle* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. *There exists a continuous function* $f : \mathbb{T} \to \mathbb{R}$ *such that Lebesgue measure on* $\mathbb{T}$ *is the unique (*T*-invariant)* f*-maximizing measure.*

In the context of Corollary 1, one would hope that Fourier analysis could be used to explicitly construct such an f. For the map $T : x \mapsto 2x \pmod 1$, maximizing measures for certain smooth functions f have been determined either experimentally or rigorously (see e.g. [Bou1, HO, Jen1]) and appear to be typically of zero entropy[5] (cf. the discussion in [Jen3, §4.3]). Roughly speaking, the more oscillatory f is, the more complicated the f-maximizing measures seem to become, in the sense that their generic points tend to have higher symbolic complexity. From the entropy point of view, Lebesgue measure is the most complicated T-invariant measure: it is the unique measure of maximal entropy. This suggests that any f as in Corollary 1 will probably be highly oscillatory.

2. **Preliminaries.** Let $C(X)$ denote the space of continuous real-valued functions on X. This is a real Banach space when equipped with the supremum norm. Let E denote the topological dual $C(X)'$, i.e., the vector space of continuous linear functionals on $C(X)$. By the Riesz representation theorem, E may be identified with the vector space of signed Borel measures on X, and we shall freely use this identification. The duality of the pair $(C(X), E)$ is given by

$$(f, \mu) \mapsto \langle f, \mu \rangle = \int f \, d\mu \,. \tag{1}$$

Let (E, w^*) denote the space E equipped with the weak* topology. By definition, this is the weakest topology such that for every $f \in C(X)$, the linear functional on E defined by $\mu \mapsto \langle f, \mu \rangle$ is continuous. This topology is locally convex, being generated by the family of semi-norms $\{p_f : f \in C(X)\}$, where $p_f(\mu) = |\langle f, \mu \rangle|$ (see [Sch, p. 48]). Since X is compact and metrizable, $C(X)$ is separable [Wal, Thm. 0.19]. Consequently [AB2, Thm. 10.7], the closed unit ball in $C(X)' = E$ is metrizable with respect to the weak* topology[6].

E is also a Riesz space (see e.g. [AB1, AB2] for background on Riesz spaces): it is an ordered vector space with respect to the (convex pointed) cone C of all positive Borel measures on X, and it is a lattice with the operations $\vee$ and $\wedge$ given by

$$(\mu \vee \nu)(A) = \sup\{\mu(B) + \nu(A \setminus B) : B \in \mathfrak{B} \,, B \subset A\} \,, \tag{2}$$

$$(\mu \wedge \nu)(A) = \inf\{\mu(B) + \nu(A \setminus B) : B \in \mathfrak{B} \,, B \subset A\} \,. \tag{3}$$

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[5]Cases where the maximizing measure is unique and of zero entropy may be considered as versions of the third law of thermodynamics, since the maximizing measure is a zero temperature limit of a certain family of equilibrium states (see e.g. [AL, CLT, CG, Jen2, JMU]).

[6]E itself is not weak* metrizable unless $C(X)$ is finite dimensional [AB2, Thm. 10.6], which is only the case if X is finite.

For $\mu \in E$, the measures $\mu^+ = \mu \vee 0$ and $\mu^- = (-\mu) \vee 0$ are positive and mutually singular, and they give the Jordan decomposition of μ, namely

$$\mu = \mu^+ - \mu^- . \tag{4}$$

The dual norm $\|\cdot\|$ on E (i.e., the one induced by the norm on $C(X)$) is known as the total variation norm (see e.g. [Rud, Ch. 6]) and can be written as

$$\|\mu\| = (\mu^+ + \mu^-)(X) . \tag{5}$$

Let E_ϕ denote the set of signed ϕ-invariant measures, i.e., those $\mu \in E$ for which $\mu(\phi_\gamma^{-1} A) = \mu(A)$ for all $\gamma \in \Gamma$, $A \in \mathfrak{B}$. A measure $\mu \in E$ belongs to E_ϕ if and only if $\langle f \circ \phi_\gamma, \mu \rangle = \langle f, \mu \rangle$ for all $f \in C(X)$, $\gamma \in \Gamma$ (see [Wal, Thm. 6.8]). Let $B_\phi(X)$ denote the closure in $C(X)$ of the vector subspace generated by the set $\{f - f \circ \phi_\gamma : f \in C(X), \gamma \in \Gamma\}$. It is easily shown that a measure $\mu \in E$ belongs to E_ϕ if and only if $\int h\, d\mu = 0$ for all $h \in B_\phi(X)$. Since E is the topological dual of $C(X)$, we deduce that E_ϕ is the topological dual of the quotient Banach space $C(X)/B_\phi(X)$.

The following characterisation of $B_\phi(X)$ will be useful[7].

Lemma 1. $B_\phi(X) = \{h \in C(X) : \langle h, \mu \rangle = 0$ *for all* $\mu \in E_\phi\}$.

Proof. Since E_ϕ is the topological dual of $C(X)/B_\phi(X)$, it follows that the topological dual of $(E_\phi, w^*) = ((C(X)/B_\phi(X))', w^*)$ is precisely $C(X)/B_\phi(X)$ (see [AB2, Thm. 9.16]). But there is another expression for the topological dual of (E_ϕ, w^*), namely

$$(E_\phi, w^*)' = C(X)/\mathrm{Ann}(E_\phi) , \tag{6}$$

where $\mathrm{Ann}(E_\phi) = \{h \in C(X) : \langle h, \mu \rangle = 0 \text{ for all } \mu \in E_\phi\}$ denotes the annihilator of E_ϕ. To verify (6), first note that by Hahn-Banach, any continuous linear functional on (E_ϕ, w^*) is the restriction of a continuous linear functional on (E, w^*). Such a functional can therefore be identified with an element of $C(X)$, which is the topological dual of (E, w^*) by [AB2, Thm. 9.16] and the Riesz representation theorem. But two elements of $C(X)$ yield the same functional on E_ϕ if and only if their difference lies in $\mathrm{Ann}(E_\phi)$, so (6) follows. Comparison of the two expressions for $(E_\phi, w^*)'$ yields the result. □

The duality of the pair $(C(X)/B_\phi(X), E_\phi)$ will be denoted by

$$(\theta, \mu) \mapsto \langle \theta, \mu \rangle ,$$

which is consistent with (1) in the sense that $\langle g, \mu \rangle = \langle \theta, \mu \rangle$ for all $g \in \theta \in C(X)/B_\phi(X)$ and $\mu \in E_\phi$, by Lemma 1.

We now review some basic properties of the set $\mathcal{M}_\phi$ of ϕ-invariant Borel probability measures. Clearly $\mathcal{M}_\phi$ is convex: if μ, ν are invariant probability measures, then so is $\lambda\mu + (1-\lambda)\nu$ for every $0 \le \lambda \le 1$. Given a convex subset K of a vector space V, a point $k \in K$ is called an *extreme point* of K if whenever $k = \lambda k_1 + (1-\lambda)k_2$ for some $k_1, k_2 \in K$ and $0 < \lambda < 1$, then $k = k_1 = k_2$. If K is contained in a hyperplane which does not contain the origin[8], it is called a *simplex* if the cone $P = \{ck : c \ge 0, k \in K\}$ defines a lattice ordering on $P - P \subset V$ (see [Phe1, p. 59]).

The following lemma details some classical facts about $\mathcal{M}_\phi$.

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[7] In certain circumstances, a more direct proof of Lemma 1 is possible (see e.g. [BJ, Lem. 3]).

[8] The assumption that K is contained in such a hyperplane is not essential, but it slightly simplifies the definition of a simplex and is satisfied in the case $K = \mathcal{M}_\phi$.

Lemma 2.
(i) $(\mathcal{M}_\phi, w^*)$ *is compact and metrizable.*
(ii) $\mathcal{M}_\phi$ *is a simplex.*
(iii) *The set of extreme points of* $\mathcal{M}_\phi$ *is precisely* $\mathcal{M}_{erg}$.

Proof. (i) This is very well known (see e.g. [Phe1, Ch. 10] and [Wal, Thm. 6.10 (i)]), but the proof is short enough to include here. The closed unit ball in E is weak* compact by the Banach-Alaoglu theorem (since E is the topological dual of a Banach space), and, as already noted, it is weak* metrizable as well. $\mathcal{M}_\phi$ is easily seen to be a weak* closed subset of this closed unit ball, so it is itself weak* compact and metrizable.
(ii) The fact that $\mathcal{M}_\phi$ is a simplex is classical, dating back at least as far as the paper of Choquet [Cho]. Since $\mathcal{M}_\phi$ lies in a hyperplane in E which does not contain the origin, it suffices to show that $E_\phi = C_\phi - C_\phi$ is a sub-lattice[9] of E. This was proved by Choquet [Cho, p. 139–14], but for completeness we give a (slightly different) proof here[10]. To verify that E_ϕ is a lattice with respect to the operations $\vee$ and $\wedge$ defined by (2) and (3), it suffices to show that if $\mu \in E_\phi$, then $\mu^+ \in E_\phi$, by [AB2, Thm. 1.2 (1), (3)]. For any $\gamma \in \Gamma$, $\mu(A) = \mu(\phi_\gamma^{-1}A) = \mu^+(\phi_\gamma^{-1}A) - \mu^-(\phi_\gamma^- A)$ for every $A \in \mathfrak{B}$, and [Rud, Corollary, p. 126] then implies that $\mu^+(\phi_\gamma^{-1}A) \geq \mu^+(A)$. But $A^c \in \mathfrak{B}$ as well, so $\mu^+((\phi_\gamma^{-1}A)^c) = \mu^+(\phi_\gamma^{-1}A^c) \geq \mu^+(A^c)$, and therefore in fact $\mu^+(\phi_\gamma^{-1}A) = \mu^+(A)$. Hence $\mu^+ \in E_\phi$.
(iii) Proofs of this well known result can be found in [Phe1, Prop. 10.4] and [Wal, Thm. 6.10 (iii)]. □

If G is a non-empty subset of a convex set K, its *convex hull* $\mathrm{co}(G)$ is the smallest convex set containing G. Its *closed convex hull* $\overline{\mathrm{co}}(G)$ is the smallest closed convex set containing G, and it equals the closure of $\mathrm{co}(G)$. A non-empty convex subset F of K is called a *face* of K if whenever $\lambda k_1 + (1-\lambda)k_2 \in F$ for some $k_1, k_2 \in K$ and $0 < \lambda < 1$, then $k_1, k_2 \in F$. We shall be particularly interested in *closed* faces. The simplest closed faces are singletons $\{k\}$, where $k \in K$ is an extreme point. The following lemma summarises certain classical properties of the closed faces of $\mathcal{M}_\phi$, which follow from the fact that it is a simplex and that $\mathcal{M}_{\mathrm{erg}}$ is its set of extreme points. Note that because $\mathcal{M}_\phi$ will always be equipped with the weak* topology, we often simply write $\mathcal{M}_\phi$ to denote $(\mathcal{M}_\phi, w^*)$.

Lemma 3.
(i) *Every closed face of* $\mathcal{M}_\phi$ *is of the form* $\overline{\mathrm{co}}(\mathcal{E})$ *for some non-empty subset* $\mathcal{E}$ *of* $\mathcal{M}_{erg}$.
(ii) *If* $\mathcal{E}$ *is a non-empty subset of* $\mathcal{M}_{erg}$ *which is closed in* $\mathcal{M}_\phi$, *then* $\overline{\mathrm{co}}(\mathcal{E})$ *is a face of* $\mathcal{M}_\phi$.
(iii) *If* $\mathcal{M}_{erg}$ *is closed in* $\mathcal{M}_\phi$, *and* $\mathcal{E}$ *is any non-empty subset of* $\mathcal{M}_{erg}$, *then* $\overline{\mathrm{co}}(\mathcal{E})$ *is a face of* $\mathcal{M}_\phi$.

Proof. (i) If $\mathcal{F}$ is a closed face of $\mathcal{M}_\phi$ then the Krein-Milman theorem implies that $\mathcal{F} = \overline{\mathrm{co}}(\mathcal{E})$, where $\mathcal{E}$ is the set of extreme points of $\mathcal{F}$. Since $\mathcal{F}$ is a face, $\mathcal{E}$ must be a subset of $\mathcal{M}_{\mathrm{erg}}$ (cf. [Alf1, Prop. 2]).
(ii) This was proved by Effros [Eff, Thm. 3.3, Cor. 3.5] (see [ER, GdR, Rog, Tay] for related discussion).

[9] In fact, E_ϕ is a Riesz subspace of E.
[10] Yet another method of proving that $\mathcal{M}_\phi$ is a simplex, based on unpublished notes of Feldman [Fel], can be found in [Phe1, Ch. 10].

(iii) This is a result of Alfsen [Alf1, Prop. 4] (see also [Alf2, Lem. II.7.18]). (A generalisation of this result (see e.g. [AA, Stø]) is that the set of extreme points of a simplex K is a closed subset if and only if $\overline{\mathrm{co}}(\cup_\alpha \mathcal{F}_\alpha)$ is a closed face for every family of closed faces $\{\mathcal{F}_\alpha\}$.) □

Let K be a convex subset of a topological vector space. A functional $l : K \to \mathbb{R}$ is *affine* if $l(\lambda k_1 + (1-\lambda)k_2) = \lambda l(k_1) + (1-\lambda)l(k_2)$ for all $k_1, k_2 \in K$ and $0 \le \lambda \le 1$. We shall be interested in affine functionals which are continuous. A face F of K is said to be *exposed* (cf. [Alf2, p. 119]) if there exists a continuous affine functional $l : K \to \mathbb{R}$ such that $l(k) = 0$ for all $k \in F$, and $l(k) > 0$ for all $k \in K \setminus F$. In particular, a (necessarily extreme) point $k \in K$ is an *exposed point* if $\{k\}$ is an exposed face. The continuity of l means that any exposed face is necessarily closed. There are simple examples of convex sets K with closed faces which are not exposed: for example, if $\mathbb{D}$ is the closed unit disc in the complex plane, and K is the convex hull of $\mathbb{D} \cup \{1+i\}$, then the extreme points 1 and i are both non-exposed.

A key result for our purposes is that if K is a compact metrizable simplex, then all of its closed faces are exposed. This was originally conjectured by Bauer [Bau, p. 121], then announced by Boboc & Cornea [BC, p. 2566] and proved by Davies [Dav, Thm. 7.4] (see also [Edw2, Prop. 4]). Since this result is crucial in proving Theorems 1–4, we provide a proof here, following closely the exposition in [FLP, Cor. 3.13] (see [Alf2, Cor. II.5.20] for an alternative approach).

Lemma 4. *Let $\mathcal{F}$ be a closed face of $\mathcal{M}_\phi$. There exists an affine functional $l : \mathcal{M}_\phi \to \mathbb{R}$, continuous in the weak* topology, such that $l(\mu) = 0$ for all $\mu \in \mathcal{F}$, and $l(\nu) > 0$ for all $\nu \in \mathcal{M}_\phi \setminus \mathcal{F}$.*

Proof. We first show that for each $\mu \in \mathcal{M}_\phi \setminus \mathcal{F}$, there is a non-negative continuous affine functional $l_\mu : \mathcal{M}_\phi \to \mathbb{R}$ such that $l_\mu|_{\mathcal{F}} \equiv 0$ and $l_\mu(\mu) > 0$. Since E is a locally convex topological vector space, a standard consequence of the Hahn-Banach theorem asserts that the two non-empty disjoint compact convex subsets $\{\mu\}$ and $\mathcal{F}$ can be separated by a continuous linear functional (see e.g. [AB2, Thm. 9.12]). Therefore, there is a continuous affine functional $\lambda : \mathcal{M}_\phi \to \mathbb{R}$ such that $\lambda(\mu) > 0$ and $\lambda|_{\mathcal{F}} \le 0$. Define $\eta : \mathcal{M}_\phi \to \mathbb{R}$ by $\eta(\nu) = \max(\lambda(\nu), 0)$, $\xi : \mathcal{M}_\phi \to \mathbb{R}$ by $\xi|_{\mathcal{F}} \equiv 0$ and $\xi|_{\mathcal{M}_\phi \setminus \mathcal{F}} \equiv \max_{m \in \mathcal{M}_\phi} |\lambda(m)|$. Since η is continuous and convex, ξ is lower semi-continuous and concave, and $\eta \le \xi$, a theorem of Edwards [Edw1] (cf. [Alf2, Thm. II.3.10]) asserts the existence of a continuous affine functional $l_\mu : \mathcal{M}_\phi \to \mathbb{R}$ such that $\eta \le l_\mu \le \xi$. In particular, l_μ is non-negative because η is, and it vanishes on $\mathcal{F}$ since both η and ξ do. Moreover, $l_\mu(\mu) \ge \eta(\mu) = \lambda(\mu) > 0$, as claimed.

Now let $\mathcal{N}$ be any closed subset of $\mathcal{M}_\phi$ which is disjoint from $\mathcal{F}$. For each $\mu \in \mathcal{N}$, define $V_\mu = \{\nu \in \mathcal{N} : l_\mu(\nu) > 0\}$. Each V_μ is open in $\mathcal{N}$, and $\mu \in V_\mu$, so $\{V_\mu\}_{\mu \in \mathcal{N}}$ is an open cover of $\mathcal{N}$. But $\mathcal{N}$ is compact, so we may choose $\mu_1, \ldots, \mu_n \in \mathcal{N}$ such that $\mathcal{N} = \bigcup_{i=1}^n V_{\mu_i}$. The functional $l_{\mathcal{N}} : \mathcal{M}_\phi \to \mathbb{R}$ defined by $l_{\mathcal{N}} = \sum_{i=1}^n l_{\mu_i}$ is continuous, affine, non-negative, and vanishes on $\mathcal{F}$ because the same is true of each l_{μ_i}. Moreover, $l_{\mathcal{N}}$ is strictly positive on $\mathcal{N}$, since for each $\nu \in \mathcal{N}$ there exists $1 \le i \le n$ such that $l_{\mu_i}(\nu) > 0$.

Now $\mathcal{F}$ is a closed subset of the metrizable space $\mathcal{M}_\phi$ and hence is a G_δ subset, so we may write $\mathcal{M}_\phi \setminus \mathcal{F} = \bigcup_{i=1}^\infty \mathcal{N}_i$, where each $\mathcal{N}_i$ is a closed subset of $\mathcal{M}_\phi$ which is disjoint from $\mathcal{F}$. Let $l_{\mathcal{N}_i}$ be as constructed above, scaled so that, say, $\max_{\nu \in \mathcal{M}_\phi} |l_{\mathcal{N}_i}(\nu)| = 2^{-i}$. The functional $l = \sum_{i=1}^\infty l_{\mathcal{N}_i}$ is then continuous, affine, non-negative, and vanishes on $\mathcal{F}$, since each $l_{\mathcal{N}_i}$ has this property. If $\nu \in \mathcal{M}_\phi \setminus \mathcal{F}$, then $\nu \in \mathcal{N}_i$ for some i, so $l_{\mathcal{N}_i}(\nu) > 0$ and therefore $l(\nu) > 0$. □

3. **Proof of Theorems.** The final ingredient in the proof of the theorems is the following result.

Proposition 1. *Suppose $l : \mathcal{M}_\phi \to \mathbb{R}$ is weak* continuous and affine. There exists $g \in C(X)$ such that*

$$l(\mu) = \langle g, \mu \rangle = \int g \, d\mu \quad \textit{for all } \mu \in \mathcal{M}_\phi \, . \tag{7}$$

Proof of Theorems. First, we prove Theorem 4. By Lemma 4, a subset $\mathcal{F}$ of $\mathcal{M}_\phi$ is a closed face of $\mathcal{M}_\phi$ if and only if there exists a weak* continuous affine functional $l : \mathcal{M}_\phi \to \mathbb{R}$ such that $l(\mu) = 0$ when $\mu \in \mathcal{F}$, and $l(\nu) > 0$ when $\nu \in \mathcal{M}_\phi \setminus \mathcal{F}$. By Proposition 1, we may write $l(\mu) = \int g \, d\mu$ for some $g \in C(X)$, so $\mathcal{F}$ is a closed face of $\mathcal{M}_\phi$ if and only if there exists a continuous function $f \, (= -g)$ such that $\int f \, d\mu = 0$ for all $\mu \in \mathcal{F}$ and $\int f \, d\nu < 0$ for all $\nu \in \mathcal{M}_\phi \setminus \mathcal{F}$. That is, $\mathcal{F}$ is a closed face of $\mathcal{M}_\phi$ if and only if $\mathcal{F} = \mathcal{M}_{\max}(f)$ for some $f \in C(X)$, so Theorem 4 is proved.

If $\mathcal{E}$ is a non-empty subset of $\mathcal{M}_{\mathrm{erg}}$, then $\overline{\mathrm{co}}(\mathcal{E})$ is a closed face of $\mathcal{M}_\phi$ provided either $\mathcal{E}$ is closed in $\mathcal{M}_\phi$ (by Lemma 3 (ii)) or $\mathcal{M}_{\mathrm{erg}}$ is closed in $\mathcal{M}_\phi$ (by Lemma 3 (iii)). In either case, Theorem 4 implies the existence of some $f \in C(X)$ for which $\mathcal{M}_{\max}(f) = \overline{\mathrm{co}}(\mathcal{E})$, so Theorems 2 and 3 are proved. Theorem 1 follows immediately from Theorem 2, since the singleton $\{\mu\}$ is a non-empty subset of $\mathcal{M}_{\mathrm{erg}}$ and is closed in $\mathcal{M}_\phi$. □

It remains to prove Proposition 1.

Proof of Proposition 1. Suppose $l : \mathcal{M}_\phi \to \mathbb{R}$ is weak* continuous and affine. Note that by Lemma 1, it suffices to find $\theta = g + B_\phi(X) \in C(X)/B_\phi(X)$ such that $l(\mu) = \langle \theta, \mu \rangle$ for all $\mu \in \mathcal{M}_\phi$.

The cone C_ϕ of positive invariant measures can be written as

$$C_\phi = \{c\mu : c \geq 0, \mu \in \mathcal{M}_\phi\} \subset E_\phi \, .$$

Each non-zero element $m \in C_\phi$ has a unique representation $m = c\mu$ for some $c > 0$ and $\mu \in \mathcal{M}_\phi$, since each measure in $\mathcal{M}_\phi$ is a *probability* measure. Therefore, we may define an extension $l_1 : C_\phi \to \mathbb{R}^+$ of $l : \mathcal{M}_\phi \to \mathbb{R}^+$ by setting $l_1(0) = 0$ and

$$l_1(m) = c \, l(\mu)$$

for $m \in C_\phi \setminus \{0\}$, where $m = c\mu$ is the unique representation mentioned above. Since l is affine, l_1 is *additive* in the sense that $l_1(\mu + \nu) = l_1(\mu) + l_1(\nu)$ for all $\mu, \nu \in C_\phi$. The functional l_1 is also weak* continuous. Its continuity at any non-zero element of C_ϕ follows from the continuity of l. To prove its continuity at zero, let $\{m_\alpha\}$ be a net in $C_\phi \setminus \{0\}$ such that $m_\alpha \to 0$ in C_ϕ. There exist unique $c_\alpha > 0$ and $\mu_\alpha \in \mathcal{M}_\phi$ such that $m_\alpha = c_\alpha \mu_\alpha$, and the fact that $m_\alpha \to 0$ in C_ϕ implies that $c_\alpha = c_\alpha \langle 1, \mu_\alpha \rangle = \langle 1, m_\alpha \rangle \to 0$ in $\mathbb{R}^+$. The compactness of $\mathcal{M}_\phi$ and continuity of l mean that $\{l(\mu_\alpha)\}$ is bounded independently of α, so $l_1(m_\alpha) = c_\alpha l(\mu_\alpha) \to 0 = l_1(0)$, as required.

We next extend l_1 to a functional $l_2 : E_\phi \to \mathbb{R}$ defined by

$$l_2(\mu) = l_1(\mu^+) - l_1(\mu^-) \, . \tag{8}$$

The additivity of l_1 means that l_2 is well-defined (see [AB2, Thm. 1.7]): if $\mu = m_1 - m_2$ for $m_1, m_2 \in C_\phi$ then it is easily checked that

$$l_1(m_1) - l_1(m_2) = l_1(\mu^+) - l_1(\mu^-) \, . \tag{9}$$

Moreover, $l_2 : E_\phi \to \mathbb{R}$ is linear [AB2, Thm. 1.7].

We wish to show that l_2 is weak* continuous. If E_ϕ is finite dimensional, then of course this is immediate from its linearity. More generally, if C_ϕ has non-empty interior in E_ϕ, then the positivity of l_2 (i.e., the fact that $l_2(C_\phi) \subset \mathbb{R}^+$) is enough to deduce its continuity [Sch, Thm. V.5.5 (i), p. 228]. However, C_ϕ need not have interior, so to deduce the continuity of l_2 we shall use the fact that $l_1 : C_\phi \to \mathbb{R}^+$ is continuous. Note that the lattice operations $\mu \mapsto \mu^+$ and $\mu \mapsto \mu^-$ are in general not continuous[11], so continuity of l_2 is not immediate from (8).

Now E_ϕ is the topological dual of the Banach space $C(X)/B_\phi(X)$, so the linear functional $l_2 : E_\phi \to \mathbb{R}$ is continuous if and only if its restriction to the closed unit ball B in $(E_\phi, \|\cdot\|)$ is, by [AB2, Thm. 10.14]. Here $\|\cdot\|$ is the dual norm on E_ϕ induced by the norm on $C(X)/B_\phi(X)$. Equivalently, $\|\cdot\|$ is the restriction to E_ϕ of the total variation norm on E, so that $\|\mu\| = (\mu^+ + \mu^-)(X)$ by (5). The linearity of l_2 means it suffices to show that $\ker(l_2) \cap B$ is weak* closed in E_ϕ, where $\ker(l_2)$ denotes the kernel of l_2.

Since B is metrizable, we need only show that if $\{\mu_i\}_{i=1}^\infty$ is a sequence in $\ker(l_2) \cap B$ which is weak* convergent to some $\mu \in E_\phi$, then in fact $\mu \in \ker(l_2) \cap B$. Now B is weak* closed by the Banach-Alaoglu theorem, so $\mu \in B$. Therefore it remains to show that $\mu \in \ker(l_2)$. Now each $\mu_i \in B$, so $\mu_i^+, \mu_i^- \in B \cap C_\phi$. But $B \cap C_\phi$ is weak* compact, so there exist $m_1, m_2 \in B \cap C_\phi$ such that

$$\mu_i^+ \to m_1 \quad \text{and} \quad \mu_i^- \to m_2 \tag{10}$$

along convergent subsequences. In particular[12],

$$\mu = m_1 - m_2 \,. \tag{11}$$

Now each $\mu_i \in \ker(l_2)$; in other words,

$$0 = l_2(\mu_i) = l_1(\mu_i^+) - l_1(\mu_i^-)\,. \tag{12}$$

But $l_1 : C_\phi \to \mathbb{R}^+$ is weak* continuous, so (10) and (12) together give

$$l_1(m_1) - l_1(m_2) = 0\,. \tag{13}$$

Combining (8), (9), (11) and (13), we deduce that

$$l_2(\mu) = l_1(\mu^+) - l_1(\mu^-) = l_1(m_1) - l_1(m_2) = 0\,,$$

so $\mu \in \ker(l_2)$, as required.

So l_2 is a weak* continuous linear functional defined on E_ϕ. As noted in §2, the topological dual of $(E_\phi, w^*) = ((C(X)/B_\phi(X))', w^*)$ is $C(X)/B_\phi(X)$. Therefore, there exists a (unique) $\theta \in C(X)/B_\phi(X)$ such that $l_2(\mu) = \langle \theta, \mu \rangle$ for all $\mu \in E_\phi$. But $l_2 : E_\phi \to \mathbb{R}$ is an extension of $l : \mathcal{M}_\phi \to \mathbb{R}^+$, so the proposition is proved. □

4. **Equilibrium states.** Given $f \in C(X)$, an *equilibrium state* for f is a measure $\mu \in \mathcal{M}_\phi$ such that $h(\mu) + \int f\, d\mu = \sup_{m \in \mathcal{M}_\phi}(h(m) + \int f\, dm)$, where $h(m)$ denotes the entropy of m (see e.g. [Rue, Wal]). The material in the preceding sections is closely related to the work of Israel & Phelps [IP, Phe2], who prove that for expansive dynamical systems, every ergodic measure is the unique equilibrium state for some continuous function f. The expansivity hypothesis guarantees the upper semi-continuity of the entropy map $\mu \mapsto h(\mu)$ on $\mathcal{M}_\phi$, and hence $\mu \mapsto h(\mu) + \int f\, d\mu$ is upper semi-continuous for any fixed $f \in C(X)$. In fact, the analysis of Israel & Phelps applies to a more general class of upper semi-continuous functionals on

[11] The lattice operations are continuous only when E_ϕ is finite dimensional (see [AB1, Thms. 5.2, 6.9]).

[12] Note that in general, it is not the case that $m_1 = \mu^+$, $m_2 = \mu^-$.

$\mathcal{M}_\phi$, including the functional $\mu \mapsto \int f\,d\mu$ considered in the present article. Being independent of entropy, this functional is continuous on $\mathcal{M}_\phi$ for arbitrary (not necessarily expansive) topological actions ϕ, so the arguments of [IP, Phe2] can be applied to maximizing measures in this generality. More precisely, our Theorem 1 can be obtained by following the proof of [Phe2, Thm. 1], whose strategy is similar to the one used here, while our Theorem 4 can be obtained from Israel & Phelps [IP] by combining their Propositions 2.1 and 3.9.

Reciprocally, some of the results in this paper have analogues in the context of equilibrium states. Notably, the following two theorems can be proved in the same way as Theorems 2 and 3.

Theorem 5. *Suppose the topological action ϕ has the property that the entropy map $\mu \mapsto h(\mu)$ is upper semi-continuous on $\mathcal{M}_\phi$. Let $\mathcal{E}$ be a non-empty subset of $\mathcal{M}_{\mathrm{erg}}$ which is weak* closed in $\mathcal{M}_\phi$. There exists a continuous function $f : X \to \mathbb{R}$ such that the set of equilibrium states for f is precisely $\overline{\mathrm{co}}(\mathcal{E})$.*

Theorem 6. *Suppose the topological action ϕ has the property that the entropy map $\mu \mapsto h(\mu)$ is upper semi-continuous on $\mathcal{M}_\phi$. Suppose that $\mathcal{M}_{\mathrm{erg}}$ is a weak* closed subset of $\mathcal{M}_\phi$. For every non-empty subset $\mathcal{E} \subset \mathcal{M}_{\mathrm{erg}}$, there exists a continuous function $f : X \to \mathbb{R}$ such that the set of equilibrium states for f is precisely $\overline{\mathrm{co}}(\mathcal{E})$.*

Theorem 5 generalises a result of Ruelle [Rue, Cor. 3.17], which asserts that for any non-empty *finite* subset $\mathcal{E} = \{\mu_1, \ldots, \mu_n\}$ of $\mathcal{M}_{\mathrm{erg}}$, there exists a continuous f such that each element of $\mathcal{E}$ is an equilibrium state for f. It follows that every element of the convex hull $\mathrm{co}(\mathcal{E}) = \overline{\mathrm{co}}(\mathcal{E})$ is also an equilibrium state for f, though the proof in [Rue] does not guarantee that these are the *only* equilibrium states for f.

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